

AROUND THE GYSIN TRIANGLE I

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ABSTRACT. In [FSV00], chap. 5, V. Voevodsky introduces the Gysin triangle associated to a closed immersion i between smooth schemes. This triangle contains the Gysin morphism associated to i but also the residue morphism.

In [Dég04] and [Dég05b], we started a study of the Gysin triangle and especially its functoriality. In this article, we complete this study by proving notably the functoriality of the Gysin morphism of a closed immersion. This allows to define a general Gysin morphism associated to a projective morphism between smooth schemes which we study further. As an application, we deduce a direct simple proof of duality for motives of projective smooth schemes.

Finally, this study concerns also the residue morphisms as formulas involving Gysin morphisms of closed immersions have their counterpart for the corresponding residue morphism. We exploit these formulas in a computation of the E_2 -term of the coniveau spectral sequence analog to that of Quillen in K -theory and deduce result on the coniveau spectral sequence associated to realization functors.

NOTATIONS AND CONVENTIONS

We fix a perfect field k . The word scheme will stands for any separated k -scheme of finite type, and we will say that a scheme is smooth when it is smooth over the base field. The category of smooth schemes is denoted by $\mathcal{S}m(k)$. Through the paper when we talk about the codimension of a closed immersion, the rank of a projective bundle or the relative dimension of a morphism, we assumed it is constant.

We let $DM_{gm}(k)$ (resp. $DM_{gm}^{eff}(k)$) be the category of geometric motives (resp. effective geometric motives) introduced in [FSV00][chap. 5]. If X is a smooth scheme, we denote by $M(X)$ the effective motive associated with X in $DM_{gm}^{eff}(k)$.

For a morphism $f : Y \rightarrow X$ of smooth schemes, we will put simply $f_* = M(f)$. Moreover for any integer r , we sometimes put $\mathbb{Z}(r) = \mathbb{Z}(r)[2r]$ in large diagrams.

INTRODUCTION

This article is an extension of previous works of the author on the Gysin triangle, [Dég04] and [Dég05b], in the setting of triangulated mixed motives. Recall that to a closed immersion $i : Z \rightarrow X$ of codimension n between smooth schemes over a perfect field k is associated a distinguished triangle

$$M(X - Z) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} M(X - Z)[1]$$

in the triangulated category $DM_{gm}(k)$. Its construction is recalled in section 1.2. The original point in the study of *op. cit.* is that the well known formulas involving the Gysin morphism i^* – for example projection formula and excess of intersection

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formula for Chow groups – corresponds also to formulas involving the residue morphism $\partial_{X,Z}$. Indeed, they fit in a general study of the functoriality of the Gysin triangle, which is recalled in 1.16.

The main technical result which we obtain here, theorem 1.24, is the compatibility of the Gysin morphism i^* with composition, but, as in the case of the projection formulas, it also gives formulas for the residue morphism. We quote it in this introduction:

Theorem. *Let X be a smooth scheme, Y (resp. Y') be a smooth closed subscheme of X of pure codimension n (resp. m). Assume $Z = Y \cap Y'$ is smooth of pure codimension d . Put $Y_0 = Y - Z$, $Y'_0 = Y' - Z$, $X_0 = X - Y \cup Y'$.*

Then the following diagram, with i, j, k, l, i' the evident closed immersions, is commutative :

$$\begin{array}{ccccc}
 M(X) & \xrightarrow{j^*} & M(Y')(m)[2m] & \xrightarrow{\partial_{X,Y'}} & M(X - Y')[1] \\
 i^* \downarrow & (1) & \downarrow k^* & (2) & \downarrow (i')^* \\
 M(Y)(n)[2n] & \xrightarrow{l^*} & M(Z)(d)[2d] & \xrightarrow{\partial_{Y,Z}} & M(Y_0)(n)[2n+1] \\
 & & \downarrow \partial_{Y',Z} & (3) & \downarrow \partial_{X_0,Y_0} \\
 & & M(Y'_0)(m)[2m+1] & \xrightarrow{-\partial_{X_0,Y'_0}} & M(X_0)[2].
 \end{array}$$

Whereas formula (2) gives the functoriality of the Gysin triangle with respect to the Gysin morphism, formula (3) is special to the residue morphism and analog to the change of variable theorem for the residue of differential forms.

We use this result to construct the Gysin morphism $f^* : M(X) \rightarrow M(Y)(d)[2d]$ of a projective morphism $f : Y \rightarrow X$ of codimension d , by considering a factorization of f into a closed immersion and the projection of a projective bundle. Indeed, in the case of a projective bundle $p : P \rightarrow X$ of rank n , the Gysin morphism $p^* : M(X) \rightarrow M(P)(-n)[-2n]$ is given by the twists of the canonical embedding through the projective bundle isomorphism (recalled in 1.7):

$$M(P) = \bigoplus_{0 \leq i \leq n} M(X)(i)[2i].$$

The key observation (prop. 2.2) in the general construction is that, for any section s of P/X , $s^*p^* = 1$. Then we derive easily the following properties of this general Gysin morphism :

(4) For any projective morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$, $(fg)^* = g^*f^*$ (prop. 2.9).

(5) Consider a cartesian square of smooth schemes $\begin{array}{ccc} T & \xrightarrow{g} & Z \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$ such that f and g are projective of the same codimension.

Then, $f^*p_* = q_*g^*$ (prop. 2.10).

(6) Consider a cartesian square of smooth schemes $\begin{array}{ccc} T & \xrightarrow{g} & Z \\ j \downarrow & & \downarrow i \\ Y & \xrightarrow{f} & X \end{array}$ such that f is projective and i is a closed immersion. Let $h : (Y - T) \rightarrow (X - Z)$ be the morphism induced by f .

Then, $h^*\partial_{X,Z} = \partial_{Y,T}g^*$ (prop. 2.12).

- (7) Let X be a smooth scheme and $f : Y \rightarrow X$ be an étale cover. Let ${}^t f$ be the finite correspondence from X to Y given by the transpose of (the graph of) f . Then $f^* = ({}^t f)_*$ (prop. 2.13).

We also mention a generalization of the formula in point (5). The reader is referred to proposition 2.11 for details. Consider the same square but assume the morphism f (resp. g) is projective of codimension n (resp. m). Let ξ be the *excess vector bundle* over T associated to the latter square, of rank $e = n - m$. Then, $f^* p_* = (c_e(\xi) \boxtimes q_*) \circ g^*$. This formula is analog to the excess intersection formula of [Ful98, 6.6(c)].

A nice application of the general Gysin morphism is the construction of the duality pairings for a smooth projective scheme X of dimension n . Let $p : X \rightarrow \text{Spec}(k)$ (resp. $\delta : X \rightarrow X \times_k X$) be the canonical projection (resp. diagonal embedding) of X/k . We obtain duality pairings (cf theorem 2.16)

$$\begin{aligned} \eta : \mathbb{Z} &\xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{\delta^*} M(X)(-n)[-2n] \otimes M(X) \\ \epsilon : M(X) \otimes M(X)(-n)[-2n] &\xrightarrow{\delta^*} M(X) \xrightarrow{p_*} \mathbb{Z}. \end{aligned}$$

which makes $M(X)(-n)[-2n]$ a *strong dual* of $M(X)$ in the sense of Dold-Puppe. This means that the functor $(M(X)(-n)[-2n] \otimes .)$ is both left and right adjoint to the functor $(. \otimes M(X))$ and implies the Poincaré duality isomorphism between motivic cohomology and motivic homology – the fundamental class is nothing else than the Gysin morphism p^* .

The remaining part studies the spectral sequence associated to the coniveau filtration of a smooth scheme for any triangulated functor $H : DM_{gm}(k)^{op} \rightarrow \mathcal{A}$, with \mathcal{A} a Grothendieck abelian category. We notably compute the E_1 -differentials in terms of morphisms *generic motives* (see section 3.2.1 for recall on generic motives and prop. 3.13 for the computation). We deduce from this an extension to our case of classical results of Bloch-Ogus (cf [BO74]).

Let us mention a nice example. Suppose k has characteristic $p > 0$. Let W be the Witt ring of k , K its fraction field. Consider a smooth scheme X . We denote by $H_{crys}^*(X/W)$ the crystalline cohomology of X defined in [Ber74]. When X is affine, we also consider the Monsky-Washnitzer cohomology $H_{MW}^*(X)$ defined in [MW68]. In the following statement, X is assumed to be proper smooth :

- (8) Let \mathcal{H}_{MW}^* be the Zariski sheaf on $\mathcal{S}m(k)$ associated with the presheaf H_{MW}^* . then $\mathcal{H}_{MW}^*(X)$ is a birational invariant of X .
 (9) There exists a spectral sequence

$$E_2^{p,q} = H_{Zar}^p(X, \mathcal{H}_{MW}^q) \Rightarrow H_{crys}^{p+q}(X/W) \otimes K$$

converging to the filtration $N^p H_{crys}^i(X/W)_K$ generated by the images of the Gysin morphisms

$$H_{crys}^{i-2q}(Y/W)_K \rightarrow H_{crys}^i(X/W)_K$$

for regular alterations of closed subschemes T of X which are of (pure) codimension $q \geq p$.

- (10) When k is separably closed, for any $p \geq 0$, $H_{Zar}^p(X, \mathcal{H}_{MW}^p) = A^p(X) \otimes K$, group of p -codimensional cycles modulo algebraic equivalence.

The key ingredient for this spectral sequence is the *rigid cohomology* of Berthelot (e.g. [Ber97]) together with its realization $H_{rig} : DM_{gm}(k)^{op} \rightarrow K - vs$ defined in [CD06]. In fact, the spectral sequence of (9) can be obtained for any cohomological realization functor H as above and any smooth scheme X (see cor. 4.7 and prop. 4.10). Property (8) for smooth proper schemes is still true in this context (*loc. cit.*). Property (9) for smooth proper schemes is true for example when H is the realization functor attached to a mixed Weil cohomology theory \mathbb{E} in the sense of [CD06] (assuming the usual property of its non positive cohomology – cor. 4.15).

We finish this introduction by mentioning a more general work of the author on the Gysin triangle in an abstract situation. However, the direct arguments used in this text, notably with the identification of motivic cohomology with higher Chow groups make it a clear and usable reference. In fact, it is used in the recent work of Barbieri-Viale and Kahn (cf [BVK08]).

The paper is organised as follows. Section 1 contains recall on the Gysin triangle together with the main technical result (theorem 1.24). In section 2, we define the Gysin morphism of any projective morphisms between smooth schemes and deduce the Poincaré duality pairing. In section 3, we recall the coniveau filtration on a smooth scheme and associate to it a pro-exact couple in the triangulated sense. The main result is a computation of the differential of this pro-exact couple in terms of morphisms of generic motives – recall on these are given in subsection 3.2.1. Lastly, section 4 studies the realizations of this pro-exact couple.

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1. THE GYSIN TRIANGLE

1.1. Relative motives.

Definition 1.1. We call closed (resp. open) pair any couple (X, Z) (resp. (X, U)) such that X is a smooth scheme and Z (resp. U) is a closed (resp. open) subscheme of X .

Let (X, Z) be an arbitrary closed pair. We will say (X, Z) is smooth if Z is smooth. For an integer n , we will say that (X, Z) has codimension n if Z is everywhere of codimension n in X .

A morphism of open or closed pair $(Y, B) \rightarrow (X, A)$ is a couple of morphisms (f, g) which fits into the commutative diagram of schemes

$$\begin{array}{ccc} B & \xrightarrow{\subset} & Y \\ g \downarrow & & \downarrow f \\ A & \xrightarrow{\subset} & X. \end{array}$$

If the pairs are closed, we require also that this diagram is cartesian on the associated topological spaces.

We add the following definitions :

- The morphism (f, g) is said to be cartesian if the above square is cartesian as a square of schemes.
- A morphism (f, g) of closed pairs is said to be excisive if f is étale and g_{red} is an isomorphism.

We will denote conventionally open pairs as fractions (X/U) .

Definition 1.2. Let (X, Z) be a closed pair. We define the relative motive $M_Z(X)$ - sometimes denoted by $M(X/X - Z)$ - associated to (X, Z) to be the class in $DM_{gm}^{eff}(k)$ of the complex

$$\dots \rightarrow 0 \rightarrow [X - Z] \rightarrow [X] \rightarrow 0 \rightarrow \dots$$

where $[X]$ is in degree 0.

Relative motives are functorial against morphisms of closed pairs. In fact, $M_Z(X)$ is functorial with respect to morphisms of the associated open pair $(X/X - Z)$. For example, if $Z \subset T$ are closed subschemes of X , we get a morphism $M_T(X) \rightarrow M_Z(X)$.

If $j : (X - Z) \rightarrow X$ denote the canonical inclusion, we obtain a canonical distinguished triangle in $DM_{gm}^{eff}(k)$:

$$(1.1) \quad M(X - Z) \xrightarrow{j^*} M(X) \rightarrow M_Z(X) \rightarrow M(X - Z)[1].$$

Remark 1.3. The relative motive in $DM_{gm}^{eff}(k)$ defined here corresponds under the canonical embedding to the relative motive in $DM_-^{eff}(k)$ defined in [Dég04][def. 2.2].

The following proposition sums up the basic properties of relative motives. It follows directly from [Dég04][1.3] using the previous remark. Note moreover that in the category $DM_{gm}^{eff}(k)$, each property is rather clear, except **(Exc)** which follows from the embedding theorem [FSV00][chap. 5, 3.2.6] of Voevodsky.

Proposition 1.4. *Let (X, Z) be a closed pair. The following properties of $M(X, Z)$ holds :*

- (1) **(Red)** Reduction : $M(X, Z) = M(X, Z_{red})$.

- (2) **(Exc)** Excision : If $(f, g) : (Y, T) \rightarrow (X, Z)$ is an excisive morphism, $(f, g)_*$ is an isomorphism.
- (3) **(MV)** Mayer-Vietoris : If $X = U \cup V$ is an open covering of X , the following triangle is distinguished :

$$\begin{array}{ccc} M(U \cap V, Z \cap U \cap V) & \xrightarrow{M(j_U) - M(j_V)} & M(U, Z \cap U) \oplus M(V, Z \cap V) \\ & \searrow \xrightarrow{M(i_U) + M(i_V)} & M(X, Z) \xrightarrow{+1} . \end{array}$$

The morphism i_U, i_V, j_U, j_V stands for the obvious cartesian morphisms of closed pairs induced by the corresponding canonical open immersions.

- (4) **(Add)** Additivity : Let Z_2 be a closed subscheme of X disjoint from $Z_1 = Z$. Then the morphism induced by the inclusions

$$M(X, Z_1 \sqcup Z_2) \rightarrow M(X, Z_1) \oplus M(X, Z_2)$$

is an isomorphism.

- (5) **(Htp)** Homotopy : Let $\pi : (\mathbb{A}_X^1, \mathbb{A}_Z^1) \rightarrow (X, Z)$ denote the cartesian morphism induced by the projection. Then π_* is an isomorphism.

1.2. Purity isomorphism.

1.5. Recall that we have an isomorphism

$H_{\mathcal{M}}^{2i}(X; \mathbb{Z}(i)) \simeq \text{Hom}_{DM_{gm}^{eff}(k)}(M(X), \mathbb{Z}(i)[2i])$, for a smooth scheme X and an integer $i \geq 0$. We will usually identify classes in motivic cohomology with morphisms in $DM_{gm}^{eff}(k)$ by this isomorphism.

Thus cup-product on motivic cohomology corresponds to a product on morphisms. Let X be a smooth scheme, $\Delta : X \rightarrow X \times_k X$ the diagonal embedding, and $f : M(X) \rightarrow \mathcal{M}$, $g : M(X) \rightarrow \mathcal{N}$ two morphisms with target a geometric motive. We define the exterior product of f and g , denoted by $f \boxtimes g$ or simply $f \boxtimes g$, as the composite

$$M(X) \xrightarrow{\Delta_*} M(X) \otimes M(X) \xrightarrow{f \otimes g} \mathcal{M} \otimes \mathcal{N}.$$

In the case where $\mathcal{M} = \mathbb{Z}(i)[2i]$, $\mathcal{N} = \mathbb{Z}(j)[2j]$, identifying the tensor product $\mathbb{Z}(i)[2i] \otimes \mathbb{Z}(j)[2j]$ with $\mathbb{Z}(i+j)[2(i+j)]$ by the canonical isomorphism, the above product corresponds exactly to the cup-product on motivic cohomology.

1.6. Recall that we have an isomorphism $H_{\mathcal{M}}^{2i}(X; \mathbb{Z}(i)) \simeq \text{CH}^i(X)$ for X a smooth scheme and i a positive integer¹. As a consequence, motivic cohomology admits Chern classes.

We thus associate to a vector bundle E on a smooth scheme X and an integer $i \geq 0$, the morphism $c_i(E) : M(X) \rightarrow \mathbb{Z}(i)[2i]$ corresponding to the i th Chern class of E .

Note that from the functoriality statement of [Dég02][8.3.4], these Chern classes are compatible with pullbacks in an obvious sense. Moreover, each relation of classical Chern classes involving intersection product corresponds to a relation of motivic Chern classes involving the above exterior product of morphisms.

1.7. We finally recall the projective bundle theorem (cf [FSV00], chap. 5, 3.5.1). Let P be a projective bundle of rank n on a smooth scheme X , λ its canonical

¹ In this setting, this isomorphism is due to Voevodsky. A detailed proof can be found in [Dég02][8.3.4].

dual line bundle and $p : P \rightarrow X$ the canonical projection. The projective bundle theorem of Voevodsky says that the morphism

$$M(P) \xrightarrow{\sum_{i \leq n} c_1(\lambda)^i \boxtimes p_*} \bigoplus_{i=0}^n M(X)(i)$$

is an isomorphism.

Thus, we can associate to P a family of split monomorphisms indexed by an integer $r \in [0, n]$ corresponding to the decomposition of its motive :

$$l_r(P) : M(X)(r)[2r] \rightarrow \oplus_{i \leq n} M(X)(i)[2i] \rightarrow M(P).$$

1.8. Consider a smooth closed pair (X, Z) . Let $N_Z X$ (resp. $B_Z X$) be the normal bundle (resp. blow-up) of (X, Z) and $P_Z X$ be the projective completion of $N_Z X$. We denote by $B_Z(\mathbb{A}_X^1)$ the blow-up of \mathbb{A}_X^1 with center $\{0\} \times Z$. It contains as a closed subscheme the trivial blow-up $\mathbb{A}_Z^1 = B_Z(\mathbb{A}_Z^1)$. We consider the closed pair $(B_Z(\mathbb{A}_X^1), \mathbb{A}_Z^1)$ over \mathbb{A}_k^1 . Its fiber over 1 is the closed pair (X, Z) and its fiber over 0 is $(B_Z X \cup P_Z X, Z)$. Thus we can consider the following deformation diagram :

$$(1.2) \quad (X, Z) \xrightarrow{\bar{\sigma}_1} (B_Z(\mathbb{A}_X^1), \mathbb{A}_Z^1) \xleftarrow{\bar{\sigma}_0} (P_Z X, Z).$$

This diagram is functorial in (X, Z) with respect to cartesian morphisms of closed pairs. Note finally that, on the closed subschemes of each closed pair, $\bar{\sigma}_0$ (resp. $\bar{\sigma}_1$) is the 0-section (resp. 1-section) of \mathbb{A}_Z^1/Z .

The existence statement in the following proposition appears already in [Dég05b, 2.2.5] but the uniqueness statement is new :

Proposition 1.9. *Let n be a natural integer.*

There exist a unique family of isomorphisms of the form

$$\mathfrak{p}_{(X,Z)} : M_Z(X) \rightarrow M(Z)(n)[2n]$$

indexed by smooth closed pairs of codimension n such that :

- (1) *for every cartesian morphism $(f, g) : (Y, T) \rightarrow (X, Z)$ of smooth closed pairs of codimension n , the following diagram is commutative :*

$$\begin{array}{ccc} M_T(Y) & \xrightarrow{(f,g)*} & M_Z(X) \\ \mathfrak{p}_{(Y,T)} \downarrow & & \downarrow \mathfrak{p}_{(X,Z)} \\ M(T)(n)[2n] & \xrightarrow{g_*(n)[2n]} & M(Z)(n)[2n]. \end{array}$$

- (2) *Let X be a smooth scheme and P the projectivization of a vector bundle E/X of rank n . Consider the pair (P, X) where X is seen as a closed subscheme through the 0-section of E/X . Then $\mathfrak{p}_{(P,X)}$ is the inverse of the following morphism*

$$M(X)(n)[2n] \xrightarrow{l_n(P)} M(P) \xrightarrow{(1)} M_X(P)$$

where (1) is the canonical split epimorphism.

Proof. Uniqueness : Consider a smooth closed pair (X, Z) of codimension n .

Applying property (1) to the deformation diagram (1.2), we obtain the commutative diagram :

$$\begin{array}{ccccc}
 M(X, Z) & \xrightarrow{\bar{\sigma}_{1*}} & M(B_Z(\mathbb{A}_X^1), \mathbb{A}_Z^1) & \xleftarrow{\bar{\sigma}_{0*}} & M(P_Z X, Z) \\
 \mathfrak{p}_{(X, Z)} \downarrow & & \mathfrak{p}_{(B_Z(\mathbb{A}_X^1), \mathbb{A}_Z^1)} \downarrow & & \downarrow \mathfrak{p}_{(P_Z X, Z)} \\
 M(Z)(n)[2n] & \xrightarrow{s_{1*}} & M(\mathbb{A}_Z^1)(n)[2n] & \xleftarrow{s_{0*}} & M(Z)(n)[2n]
 \end{array}$$

Using homotopy invariance, s_{0*} and s_{1*} are isomorphisms. Thus in this diagram, all morphisms are isomorphisms. Now, the second property of the purity isomorphisms determines uniquely $\mathfrak{p}_{(P_Z X, Z)}$, thus $\mathfrak{p}_{(X, Z)}$ is also uniquely determined.

For the existence part, we refer the reader to [Dég05b], section 2.2. □

For a smooth pair (X, Z) , we will call $\mathfrak{p}_{(X, Z)}$ the *purity isomorphism*.

Remark 1.10. The second point of the above proposition appears as a normalization condition. It will be reinforced latter (cf rem. 2.3).

Using the purity isomorphism introduced previously, we introduce the following definition :

Definition 1.11. Let (X, Z) be a closed pair such that Z is smooth and of codimension n in X . Denote by j (resp. i) the open immersion $(X - Z) \rightarrow X$ (resp. closed immersion $Z \rightarrow X$).

Using the purity isomorphism $\mathfrak{p}_{(X, Z)}$, we deduce from the distinguished triangle (1.1) the following distinguished triangle in $DM_{gm}^{eff}(k)$, called the Gysin triangle of (X, Z)

$$M(X - Z) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X, Z}} M(X - Z)[1].$$

The morphisms $\partial_{(X, Z)}$ (resp. i^*) is called the *residue* (resp. *Gysin morphism*) associated to (X, Z) (resp. i). Sometimes we use the notation $\partial_i = \partial_{(X, Z)}$.

Example 1.12. Consider a smooth scheme X and a vector bundle E/X of rank n . Let P be the projective completion of E , λ be its canonical dual invertible sheaf, and $p : P \rightarrow X$ be its canonical projection. Consider the canonical section $i : X \rightarrow P$ of P/X .

Consider the Thom class of E in $CH^n(P)$ as the class

$$t(E) = \sum_{i=0}^n p^*(c_{n-i}(E)) \cdot c_1(\lambda)^i.$$

It corresponds to a morphism $\mathfrak{t}(E) : M(P) \rightarrow \mathbb{Z}(n)[2n]$. The restriction of $t(E)$ to $P - X$ is zero. Thus the morphism

$$\mathfrak{t}(E) \boxtimes_{Pp_*} : M(P) \rightarrow M(X)(n)[2n]$$

canonically induces a morphism $\epsilon_P : M_X(P) \rightarrow M(X)(n)[2n]$. From the point (2) in proposition 1.9, it follows readily that $\epsilon_P \circ \mathfrak{p}_{(P, Z)} = 1$.

As a consequence, we obtain the formula

$$i^* = \mathfrak{t}(E) \boxtimes_{Pp_*}$$

which is the analog of the well know fact that $i_*(1) = t(E)$ in the Chow group of P .

Remark 1.13. Our Gysin triangle agrees with that of [FSV00], chap. 5, prop. 3.5.4. Indeed, in the proof of 3.5.4, Voevodsky constructed an isomorphism called $\alpha_{(X,Z)}$ and used it as we use the purity isomorphism to construct his triangle. It is not hard to check that this isomorphism satisfies the two conditions of proposition 1.9, and thus coincides with the purity isomorphism from the uniqueness statement.

1.3. Base change formulas. This subsection is devoted to recall some results we have previously obtained in [Dég04] and [Dég05b] about the following type of morphism :

Definition 1.14. Let (X, Z) (resp. (Y, T)) be a smooth closed pair of codimension n (resp. m). Let $(f, g) : (Y, T) \rightarrow (X, Z)$ be a morphism of closed pairs.

We define the morphism $(f, g)_!$ as the following composite :

$$M(T)(m)[2m] \xrightarrow{\mathbf{p}_{(Y,T)}^{-1}} M(Y, T) \xrightarrow{(f,g)_*} M(X, Z) \xrightarrow{\mathbf{p}_{(X,Z)}} M(Z)(n)[2n].$$

In the situation of this definition, let $i : Z \rightarrow X$ and $k : T \rightarrow Y$ be the obvious closed embeddings, and $h : Y - T \rightarrow X - Z$ be the restriction of f . With the definition above, we obtain the following commutative diagram :

$$\begin{array}{ccccccc} M(Y - T) & \longrightarrow & M(Y) & \xrightarrow{j^*} & M(T)(m)[2m] & \xrightarrow{\partial_{Y,T}} & M(Y - T)[1] \\ \downarrow & & \downarrow f_* & \text{(1)} & \downarrow (f,g)_! & \text{(2)} & \downarrow h_* \\ M(X - Z) & \longrightarrow & M(X) & \xrightarrow{i^*} & M(Z)(n)[2n] & \xrightarrow{\partial_{X,Z}} & M(X - Z)[1] \end{array}$$

The commutativity of square (1) corresponds to a refined projection formula. The word refined is inspired by the terminology “refined Gysin morphism” of Fulton in [Ful98]. By contrast, the commutativity of square (2) is concerned with higher Chow groups and is a phenomena of mixed motives.

Remark 1.15. Using properties (Exc) and (Add) of proposition 1.4, one can see easily that the study of the morphism $(f, g)_!$ can be reduced to the case where Z and T are integral.

Let T and T' be closed subschemes of a scheme Y , \mathcal{J} and \mathcal{J}' be their respective defining ideals and $i : T \rightarrow T'$ be a closed immersion. We will say that i is a *thickening of order r* if $\mathcal{J}' = \mathcal{J}^r$. We recall to the reader the following formulas obtained in [Dég04, 3.1, 3.3] :

Proposition 1.16. *Let (X, Z) and (Y, T) be smooth closed pairs of codimension n and m respectively. Let $(f, g) : (Y, T) \rightarrow (X, Z)$ be a morphism of closed pairs.*

- (1) (Transversal case) *If (f, g) is cartesian and $n = m$, then $(f, g)_! = g_*(n)[2n]$.*
- (2) (Excess intersection) *If (f, g) is cartesian, we put $e = n - m$ and $\xi = g^*N_Z X/N_T Y$. Then $(f, g)_! = \mathbf{c}_e(\xi) \boxtimes g_*((m))$.*
- (3) (Ramification case) *If $n = m = 1$ and the canonical closed immersion $T \rightarrow Z \times_X Y$ is an exact thickening of order r , then $(f, g)_! = r.g_*(1)[2]$.*

Remark 1.17. In the forthcoming article [Dég07, 4.23], the case (3) will be generalized to any codimension $n = m$. In the general case, the integer r is simply the geometric multiplicity of $Z \times_X Y$.

As an application of the first case of this proposition, we remark that we obtain a projection formula for the Gysin morphism :

Corollary 1.18. *Let (X, Z) be a smooth pair of codimension n , and let $i : Z \rightarrow X$ be the corresponding closed immersion.*

Then, $(1_Z \boxtimes_Z i_) \circ i^* = i^* \boxtimes_X 1_X : M(X) \rightarrow M(Z) \otimes M(X)(n)[2n]$.*

PROOF : Just apply the above formula for the cartesian transversal morphism $(X, Z) \rightarrow (X \times X, Z \times X)$ induced by the diagonal embedding of X . The only thing left to check is that $(i \times 1_X)^* = i^* \otimes 1$, which was done in [Dég05b], prop. 2.6.1. \square

Remark 1.19. In the above statement, we have loosely identified the motive $M(Z) \otimes M(X)(n)[2n]$ with $(M(Z)(n)[2n]) \otimes M(X)$ through the canonical isomorphism. This will not have any consequences in the present article. On the contrary in [Dég05b], we must be attentive to this isomorphism which may result in a change of sign (cf remark 2.6.2 of *loc. cit.*).

1.20. Let (X, Z) be a smooth closed pair of codimension n . Assume the canonical immersion $i : Z \rightarrow X$ admits a retraction $p : X \rightarrow Z$.

According to [Gro58], we define the fundamental class of Z in X as the class $\eta_X(Z) = i_*(1)$ in $H_{\mathcal{M}}^{2n}(X; \mathbb{Z}(n))$, where i_* is induced by our Gysin morphism $i^* : M(X) \rightarrow M(Z)(n)[2n]$.

If $\pi_Z : Z \rightarrow \text{Spec}(k)$ denotes the canonical projection, then the class $\eta_X(Z)$ corresponds to the morphism

$$\pi_{Z*} i^* : M(X) \rightarrow \mathbb{Z}(n)[2n].$$

Moreover, the projection formula 1.18 allows to write

$$i^* = (\pi_{Z*} \boxtimes_Z p_* i_*) \circ i^* = \eta_X(Z) \boxtimes_P p_*$$

which is the analog of the well known formula for cycles.

As a particular case, when $X = \mathbb{P}(E \oplus 1)$ is a projective bundle over Z , we obtain $\eta_X(Z) = t(E)$ (cf example 1.12).

Remark 1.21. Let (X, Z) be a smooth closed pair of codimension n , $i : Z \rightarrow X$ the corresponding closed immersion. The Gysin morphism i^* that we have defined on motives induces a pushout on motivic cohomology $H_{\mathcal{M}}^{2s}(Z; \mathbb{Z}(s)) \rightarrow H_{\mathcal{M}}^{2(s+n)}(X; \mathbb{Z}(s+n))$ which corresponds under the canonical isomorphism to a pushout on Chow groups $i_* : CH^s(Z) \rightarrow CH^{s+n}(X)$.

This pushout is the usual proper pushout on Chow groups. In the case where i is the canonical section a projective bundle $\mathbb{P}(E \oplus 1)/Z$, this follows from the computation above. In the general case, we can use the deformation diagram (1.2) to obtain a commutative diagram involving either the usual proper pushout or the pushout defined by our Gysin morphism (according to prop. 1.16) on vertical maps

$$\begin{array}{ccccc} CH^*(Z) & \xleftarrow{\bar{\sigma}_1^*} & CH^*(\mathbb{A}_Z^1) & \xrightarrow{\bar{\sigma}_0^*} & CH^*(Z) \\ \downarrow & & \downarrow & & \downarrow \\ CH^*(X) & \xleftarrow{s_1^*} & CH^*(B_Z(\mathbb{A}_X^1)) & \xrightarrow{s_0^*} & CH^*(P_Z X) \end{array}$$

The conclusion follows from the fact $\bar{\sigma}_1$ has a retraction given by the canonical projection $B_Z(\mathbb{A}_X^1) \rightarrow X$ and by functoriality of the canonical isomorphism $H_{\mathcal{M}}^{2s}(\cdot; \mathbb{Z}(s)) \simeq CH^s(\cdot)$ with respect to pullbacks.

As a consequence, in the general situation of 1.20, the fundamental class $\eta_X(Z)$ corresponds to the usual cycle class of Z in $CH^n(X)$.

1.4. Composition of Gysin triangles. We first establish lemmas needed for the main theorem. Remark that, using the projection formula in the transversal case (cf 1.16) and the compatibility of Chern classes with pullbacks, we obtain easily the following result, whose proof is left to the reader :

Lemma 1.22. *Let (Y, Z) be a smooth pair of codimension m and P/Y be a projective bundle of dimension n . We put $V = Y - Z$ and consider the following cartesian squares :*

$$\begin{array}{ccccc} P_V & \xrightarrow{\nu} & P & \xleftarrow{\iota} & P_Z \\ p_V \downarrow & & p \downarrow & & \downarrow p_Z \\ V & \xrightarrow{j} & Y & \xleftarrow{i} & Z \end{array}$$

Finally, we consider the canonical line bundle λ (resp. λ_V, λ_Z) on P (resp. P_V, P_Z).

Then, for all integer $r \in [0, n]$, the following diagram is commutative

$$\begin{array}{ccccccc} M(P_V) & \xrightarrow{\nu_*} & M(P) & \xrightarrow{\iota^*} & M(P_Z)((m)) & \xrightarrow{\partial_i} & M(P_V)[1] \\ \downarrow \epsilon_1(\lambda_V)^* \boxtimes p_{V*} & & \downarrow \epsilon_1(\lambda)^* \boxtimes p_* & & \downarrow \epsilon_1(\lambda_Z)^* \boxtimes p_{Z*} & & \downarrow \epsilon_1(\lambda_V)^* \boxtimes p_{V*}[1] \\ M(V)((r)) & \xrightarrow{j_*} & M(Y)((r)) & \xrightarrow{i^*} & M(Z)((r+m)) & \xrightarrow{\partial_i} & M(V)((r))[1]. \end{array}$$

The next lemma will be in fact the crucial case in the proof of the next theorem.

Lemma 1.23. *Let Z be a smooth scheme, E/Z and E'/Z be two vector bundles of respective ranks n and m .*

Put $Q = \mathbb{P}(E \oplus 1)$ and $Q' = \mathbb{P}(E' \oplus 1)$, $P = Q \times_Z Q'$ and consider the canonical immersions $i : Z \rightarrow Q$, $j : Q \rightarrow P$ and $k : Z \rightarrow P$.

Then $k^ = i^* j^*$.*

Proof. Let p_Z and p_Q be the respective structural morphisms of the k -schemes Z and Q and consider the canonical projections $q : Q \rightarrow Z$, $q' : P \rightarrow Q$ and $p : P \rightarrow Z$.

Following paragraph 1.20, we obtain

$$i^* = \eta_Q(Z) \boxtimes_{Qq_*} q'_*, j^* = \eta_P(Q) \boxtimes_{Pq'_*} q'_*, k^* = \eta_P(Z) \boxtimes_{Pp_*} p_*.$$

Taking into account that $\eta_Q(Z) \circ q'_* = \eta_P(Q')$ according to the transversal case of proposition 1.16, we obtain :

$$i^* j^* = \eta_P(Q) \boxtimes_P \eta_P(Q') \boxtimes_{Pp_*} p_*.$$

and we are reduced to prove the relation $\eta_P(Z) = \eta_P(Q) \boxtimes_P \eta_P(Q')$. But this relation considered in the Chow group $CH^{n+m}(P)$ is obvious as Q and Q' meet properly in P and the conclusion follows from 1.21. \square

Theorem 1.24. *Consider a cartesian square of smooth schemes*

$$\begin{array}{ccc} Z & \xrightarrow{k} & Y' \\ l \downarrow & & \downarrow j \\ Y & \xrightarrow{i} & X \end{array}$$

such that i, j, k, l are closed immersions of respective pure codimensions n, m, s, t . We put $d = n + t = m + s$ and consider the induced closed immersion $h : Y - T \rightarrow X - Z$.

Then the following diagram is the commutative :

$$\begin{array}{ccccc}
 M(X) & \xrightarrow{j^*} & M(Y')((m)) & \xrightarrow{\partial_j} & M(X - Y') [1] \\
 i^* \downarrow & (1) & \downarrow k^* & (2) & \downarrow (i')^* \\
 M(Y)((n)) & \xrightarrow{l^*} & M(Z)((d)) & \xrightarrow{\partial_l} & M(Y - Z)((n)) [1] \\
 & & \downarrow \partial_k & (3) & \downarrow \partial_{i'} \\
 & & M(Y - Z)((m)) [1] & \xrightarrow{-\partial_{j'}} & M(X - Y \cup Y') [2]
 \end{array}$$

Proof. We will call simply smooth triple (X, Y, Y') the data of three smooth schemes X, Y, Y' such that Y' and Y are smooth closed subscheme of X . As for closed pairs, these smooth triples form a category with morphism the evident commutative diagram which we require to be formed by two cartesian squares.

To such a triple, we associate a geometric motive $M(X, Y, Y')$ as the cone of the canonical map of complexes of $\mathcal{S}m^{cor}(k)$

$$\begin{array}{ccccc}
 \cdots & \rightarrow & [X - Y \cup Y'] & \rightarrow & [X - Y'] \rightarrow \cdots \\
 & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & [X] & \longrightarrow & [X - Y] \rightarrow \cdots
 \end{array}$$

This motive is evidently functorial with respect to morphisms of smooth triples. Sometimes, it will be meaningful to write it $M\left(\frac{X/X-Y}{X-Y'/X-Y \cup Y'}\right)$. By definition, it fits into the following diagram :

$$\begin{array}{ccccccc}
 (\mathcal{D}) : M(X - \Omega) & \rightarrow & M(X - Y) & \rightarrow & M\left(\frac{X-Y}{X-\Omega}\right) & \rightarrow & M(X - \Omega) [1] \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 M(X - Y') & \rightarrow & M(X) & \rightarrow & M\left(\frac{X}{X-Y'}\right) & \rightarrow & M(X - Y') [1] \\
 \downarrow & & \downarrow & (1) & \downarrow & (2) & \downarrow \\
 M\left(\frac{X-Y'}{X-\Omega}\right) & \rightarrow & M\left(\frac{X}{X-Y}\right) & \rightarrow & M\left(\frac{X/X-Y}{X-Y'/X-\Omega}\right) & \rightarrow & M\left(\frac{X-Y'}{X-\Omega}\right) [1] \\
 \downarrow & & \downarrow & & \downarrow & (3) & \downarrow \\
 M(X - \Omega) [1] & \rightarrow & M(X - Y) [1] & \rightarrow & M\left(\frac{X-Y}{X-\Omega}\right) [1] & \rightarrow & M(X - \Omega) [2],
 \end{array}$$

where $\Omega = Y \cup Y'$. In this diagram, every square is commutative except square (3) which is anticommutative due to the fact the permutation isomorphism on $\mathbb{Z}[1] \otimes \mathbb{Z}[1]$ is equal to -1 . Moreover, any line or row of this diagram formed a distinguished triangle.

With the hypothesis of the theorem, the proof will consist in constructing a purity isomorphism $\mathbf{p}_{(X,Y,Y')} : M(X, Y, Y') \rightarrow M(Z)(d)[2d]$ which satisfies the following properties :

- (i) *Functoriality* : The morphism $\mathbf{p}_{(X,Y,Y')}$ is functorial with respect the cartesian morphisms which are transversals to Y, Y' and Z .

(ii) *Symmetry* : The following diagram is commutative :

$$\begin{array}{ccc} M(X, Y, Y') & \xrightarrow{\quad\quad\quad} & M(X, Y', Y) \\ & \searrow \mathfrak{p}_{(X, Y, Y')} \quad \swarrow \mathfrak{p}_{(X, Y', Y)} & \\ & M(Z)(d)[2d] & \end{array}$$

where the horizontal map is the canonical isomorphism.

(iii) *Compatibility* : The following diagram is commutative :

$$\begin{array}{ccccccc} M\left(\frac{X-Y'}{X-\Omega}\right) & \longrightarrow & M\left(\frac{X}{X-Y}\right) & \longrightarrow & M(X, Y, Y') & \longrightarrow & M\left(\frac{X-Y'}{X-\Omega}\right)[1] \\ \downarrow \mathfrak{p}_{(X-Y', Y-Z)} & & \downarrow \mathfrak{p}_{(X, Y)} & & \downarrow \mathfrak{p}_{(X, Y, Y')} & & \downarrow \mathfrak{p}_{(X-Y', Y-Z)}[1] \\ M(Y-Z)((n)) & \longrightarrow & M(Y)((n)) & \xrightarrow{j^*} & M(Z)((d)) & \xrightarrow{\partial_j} & M(Y-Z)((n))[1] \end{array}$$

With this isomorphism, we can deduce the three relations of the theorem by considering squares (1), (2), (3) in the above diagram when we apply the evident purity isomorphism where it belongs.

We then are reduced to construct the isomorphism and to prove the above relations. The second relation is the most difficult one because we have to show that two isomorphisms in a triangulated category are equal. This forces us to be very precise in the construction of the isomorphism.

Construction of the purity isomorphism for smooth triples :

Consider the deformation diagram (1.2) for the closed pair (X, Y) and put $B = B_Y(\mathbb{A}_X^1)$, $P = P_Y X$. Put also $(U, V) = (X - Y', Y - Z)$, $B_U = B \times_X U$ and $P_V = P \times_Y V$. By functoriality of the deformation diagram and relative motives we obtain the following morphisms of distinguished triangles :

$$\begin{array}{ccccccc} M(U, V) & \longrightarrow & M(X, Y) & \longrightarrow & M\left(\frac{X/X-Y}{U/U-V}\right) & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ M(B_U, \mathbb{A}_U^1) & \longrightarrow & M(B, \mathbb{A}_Y^1) & \longrightarrow & M\left(\frac{B/B-\mathbb{A}_Y^1}{B_U/B_U-\mathbb{A}_V^1}\right) & \xrightarrow{+1} & \\ \uparrow & & \uparrow & & \uparrow & & \\ M(P_V, V) & \longrightarrow & M(P, Y) & \longrightarrow & M\left(\frac{P/P-Y}{P_V/P_V-V}\right) & \xrightarrow{+1} & \end{array}$$

The first stage of vertical morphisms is induced by the 1-section of B (resp. B_V) over \mathbb{A}_k^1 , and the second through its 0-section. Thus, they all are isomorphisms in $DM_{gm}^{eff}(k)$. The last stage is induced by forgetting some denominators.

Now, using lemma 1.22 with $P = \mathbb{P}(N_Y X \oplus 1)$, we can finally consider the following morphism of distinguished triangles :

$$\begin{array}{ccccccc}
 M(P_V, V) & \longrightarrow & M(P, Y) & \longrightarrow & M\left(\frac{P/P-Y}{P_V/P_V-V}\right) & \xrightarrow{+1} & \\
 \uparrow & & \uparrow & & \uparrow & & \\
 M(P_V) & \longrightarrow & M(P) & \longrightarrow & M\left(\frac{P}{P_V}\right) & \xrightarrow{+1} & \\
 \parallel & & \parallel & & \uparrow \mathfrak{p}_{(P, P_Z)}^{-1} & & \\
 M(P_V) & \longrightarrow & M(P) & \longrightarrow & M(P_Z)((s)) & \xrightarrow{+1} & \\
 \uparrow \iota_n(P_V) & & \uparrow \iota_n(P) & & \uparrow \iota_n(P_Z) & & \\
 M(Y-Z)((n)) & \longrightarrow & M(Y)((n)) & \longrightarrow & M(Z)((d)) & \xrightarrow{+1} &
 \end{array}$$

The triangle on the bottom is obtained by tensoring the Gysin triangle of the pair (Y, Z) with $\mathbb{Z}(n)[2n]$. From proposition 1.9, the first two of the vertical composite arrows are isomorphisms, so the last one is also an isomorphism.

If we compose (vertically) the two previous diagrams, we finally obtain the following isomorphism of triangles :

$$\begin{array}{ccccccc}
 M(U, V) & \longrightarrow & M(X, Y) & \longrightarrow & M(X, Y, Y') & \longrightarrow & M(U, V)[1] \\
 \downarrow \mathfrak{p}_{(X-Y', Y-Z)} & & \downarrow \mathfrak{p}_{(X, Y)} & & \downarrow (*) & & \downarrow \\
 M(Y-Z)((n)) & \longrightarrow & M(Y)((n)) & \xrightarrow{j^*} & M(Z)((d)) & \xrightarrow{\partial_j} & M(Y-Z)((n))[1].
 \end{array}$$

Thus we can define $\mathfrak{p}_{(X, Y, Z)}$ as the morphism labelled $(*)$ in the previous diagram so that property (iii) follows from the construction. The functoriality property (i) follows easily from the functoriality of the deformation diagram.

The remaining relation

To conclude it remains only to prove the symmetry property. First of all, we remark that the above construction implies immediately the commutativity of the following diagram :

$$\begin{array}{ccc}
 M\left(\frac{X/X-Y}{X-Y/X-Y \cup Y'}\right) & \longrightarrow & M\left(\frac{X/X-Y}{X-Z/X-Y}\right) \\
 \searrow \mathfrak{p}_{(X, Y, Y')} & & \swarrow \mathfrak{p}_{(X, Y, Z)} \\
 & M(Z)((d)), &
 \end{array}$$

where the horizontal map is induced by the evident open immersions.

Thus, it will be sufficient to prove the commutativity of the following diagram :

$$\begin{array}{ccc}
 M\left(\frac{X}{X-Z}\right) & \xrightarrow{\alpha_{X, Y, Z}} & M\left(\frac{X/X-Y}{X-Z/X-Y}\right) \\
 \searrow \mathfrak{p}_{(X, Z)} & & \swarrow \mathfrak{p}_{(X, Y, Z)} \\
 & M(Z)((n+m)), &
 \end{array}$$

where $\alpha_{X, Y, Z}$ denotes the canonical isomorphism.

From now on, we consider only the smooth triples (X, Y, Z) such that Z is a closed subscheme of Y . Using the functoriality of $\mathfrak{p}_{(X, Y, Z)}$, we remark that diagram $(**)$ is natural with respect to morphisms $f : X' \rightarrow X$ which are transversal with Y and Z .

Consider the notations of paragraph 1.8 and put $D_Z X = B_Z(\mathbb{A}_X^1)$ for short. We will expand these notations as follows :

$$D(X, Z) = D_Z X, \quad B(X, Z) = B_Z X, \quad P(X, Z) = P_Z X.$$

To (X, Y, Z) , we associate the evident closed pair $(D_Z X, D_Z X|_Y)$ and the *double deformation space*

$$D(X, Y, Z) = D(D_Z X, D_Z X|_Y).$$

This latter scheme is in fact fibered over \mathbb{A}_Z^2 . The fiber over $(1, 1)$ is X and the fiber over $(0, 0)$ is $B(B_Z X \cup P_Z X, B_Z X|_Y \cup P_Z X|_Y)$. In particular, the $(0, 0)$ -fiber contains the scheme $P(P_Z X, P_Z Y)$.

$$\text{We now put } \begin{cases} D = D(X, Y, Z), & P = P(P_Z X, P_Z Y) \\ D' = D(Y, Y, Z), & Q = P_Z Y. \end{cases}$$

Remark also that $D(Z, Z, Z) = \mathbb{A}_Z^2$ and that² $P = Q \times_Z Q'$ where $Q' = P_Y X|_Z$. From the description of the fibers of D given above, we obtain a deformation diagram of smooth triples :

$$(X, Y, Z) \rightarrow (D, D', \mathbb{A}_Z^2) \leftarrow (P, Q, Z).$$

Note that these morphisms are on the smaller closed subscheme the $(0, 0)$ -section and $(1, 1)$ -section of \mathbb{A}_Z^2 over Z , denoted respectively by s_1 and s_0 . Now we apply these morphisms to diagram (*) obtaining the following commutative diagram :

$$\begin{array}{ccccc} M_Z(X) & \xrightarrow{\quad} & M_{\mathbb{A}_Z^2}(D) & \xleftarrow{\quad} & M_Z(P) \\ \downarrow \text{p}_{(X,Z)} & \searrow \alpha_{X,Y,Z} & \downarrow \text{p}_{(D,\mathbb{A}_Z^2)} & \searrow \alpha_{P,Q,Z} & \downarrow \text{p}_{(P,Z)} \\ & M(X, Y, Z) & \xrightarrow{\quad} & M(D, D', \mathbb{A}_Z^2) & \xleftarrow{\quad} & M(P, Q, Z) \\ \downarrow \text{p}_{(X,Y,Z)} & \swarrow & \downarrow \text{p}_{(D,D',Z)} & \swarrow & \downarrow \text{p}_{(P,Q,Z)} \\ M(Z)((n+m)) & \xrightarrow{s_{1*}((n+m))} & M(\mathbb{A}_Z^2)((n+m)) & \xleftarrow{s_{0*}((n+d))} & M(Z)((n+m)). \end{array}$$

The square parts of this prism are commutative. As morphisms s_{1*} and s_{0*} are isomorphisms, the commutativity of the triangle on the left is equivalent to the commutativity of the right one.

Thus, we are reduced to the case of the smooth triple (P, Q, Z) . Now, using the canonical split epimorphism $M(P) \rightarrow M_Z(P)$, we are reduced to prove the commutativity of the diagram :

$$\begin{array}{ccc} M(P) & \xrightarrow{\quad} & M\left(\frac{P/P-Q}{P-Z/P-Q}\right) \\ i^* \downarrow & & \swarrow \text{p}_{(P,Q,Z)} \\ M(Z)((d)) & & \end{array}$$

where $i : Z \rightarrow P$ denotes the canonical closed immersion.

Using the property (iii) of the isomorphism $\text{p}_{(P,Q,Z)}$, we are finally reduced to prove the commutativity of the triangle

$$\begin{array}{ccc} M(P) & \xrightarrow{j^*} & M(Q)((n)) \\ i^* \searrow & & \swarrow k^* \\ & M(Z)((d)) & \end{array}$$

where j and k denotes the evident closed embeddings. This is lemma 1.23. \square

²This is equivalent to the fact $N(N_Z X, N_Z Y) = N_Z Y \oplus N_Y X|_Z$

As a corollary, we get functoriality of the Gysin morphism of a closed immersion :

Corollary 1.25. *Let $Z \xrightarrow{l} Y \xrightarrow{i} X$ be closed immersion between smooth schemes such that i is of pure codimension n .*

Then, $l^ \circ i^* = (i \circ l)^*$.*

2. GYSIN MORPHISM

In this section, motives are considered in the category $DM_{gm}(k)$.

2.1. Construction.

2.1.1. Preliminaries.

Lemma 2.1. *Let X be a smooth scheme, P/X and Q/X be projective bundles of respective dimension n and m . We consider λ_P (resp. λ_Q) the canonical dual invertible sheaf on P (resp. Q) and λ'_P (resp. λ'_Q) its pullback on $P \times_X Q$. Let $p : P \times_X Q \rightarrow X$ be the canonical projection.*

Then, the morphism $\sum_{0 \leq i \leq n, 0 \leq j \leq m} c_1(\lambda'_P)^i \boxtimes c_1(\lambda'_Q)^j \boxtimes p_ :$*

$$M(P \times_X Q) \longrightarrow \bigoplus_{i,j} M(X)(i+j)[2(i+j)]$$

is an isomorphism.

Proof. Let σ be the above morphism. As σ is compatible with pullback, we can suppose using the Mayer-Vietoris triangle that P and Q are trivializable projective bundles. Using the invariance of σ under automorphisms of P or Q , we can assume that P and Q are trivial projective bundles. From the definition of σ , we are reduced to the case $X = \text{Spec}(k)$. Then, σ is just the tensor product of the two projective bundle isomorphisms (cf paragraph 1.7) for P and Q . \square

The following proposition is the key point in the definition of the Gysin morphism for a projective morphism.

Proposition 2.2. *Let X be a smooth scheme, P/X a projective bundle of rank n and $s : X \rightarrow P$ a section of the canonical projection p .*

Then, the composite $M(X)((n)) \xrightarrow{l_n(P)} M(P) \xrightarrow{s^} M(X)((n))$ is the identity.*

Proof. From paragraph 1.20, we obtain $s^* = \eta_P(Z) \boxtimes p_*$.

Consider the basis $1, c_1, \dots, c_1^n$ of the $CH^*(X)$ -module $CH^*(P)$, where c_1 is the first Chern class of the canonical line bundle. From the relation $p_* s_*(1) = 1$ in $CH^*(X)$, we obtain that the coefficient of c_1^n in $\eta_P(Z)$ relative to this basis is 1. This concludes by definition of $l_n(P)$ (cf paragraph 1.7). \square

Remark 2.3. As a corollary, we obtain the following reinforcement of proposition 1.9, more precisely of the normalisation condition for the purity isomorphism :

Let P be a projective bundle of rank n over a smooth scheme X , and $s : X \rightarrow P$ be a section of P/X .

Then, the purity isomorphism $\mathfrak{p}_{(P,s(X))}$ is the reciprocal isomorphism of the composition

$$M(X)((n)) \xrightarrow{l_n(P)} M(P) \xrightarrow{(1)} M(P, s(X))$$

where (1) is the canonical (split) epimorphism.

2.1.2. *Gysin morphism of a projection.* The following definition will be a particular case of definition 2.7.

Definition 2.4. Let X be a smooth scheme, P a projective bundle of rank n over X and $p : P \rightarrow X$ be the canonical projection.

We let $p^* = \iota_n(P)(-n)[-2n] : M(X) \rightarrow M(P)(-n)[-2n]$ and call it the Gysin morphism of p .

Lemma 2.5. Let P, Q be projective bundles over a smooth scheme X of respective rank n, m . Consider the following projections :

$$\begin{array}{ccccc} & & q' & & \\ & & \nearrow & & \\ P \times_X Q & & P & \xrightarrow{p} & X \\ & & \searrow & & \\ & & Q & \xrightarrow{q} & \\ & p' & & & \end{array}$$

Then, the following diagram is commutative :

$$\begin{array}{ccccc} & & M(P)((m)) & \xrightarrow{q'^*} & \\ & p^* & \nearrow & & \\ M(X) & & & & M(P \times_X Q)((n+m)) \\ & q^* & \searrow & p'^* & \\ & & M(Q)((n)) & & \end{array}$$

Proof. Indeed, using the compatibility of the motivic Chern class with pullback, we see that both edge morphisms in the previous diagram are equal to the composite

$$M(X)((n+m)) \rightarrow \bigoplus_{i \leq n, j \leq m} M(X)((i+j)) \rightarrow M(P \times_X Q),$$

where the first arrow is the obvious split monomorphism and the second arrow is the reciprocal isomorphism of the one constructed in lemma 2.1. \square

2.1.3. *General case.* The following lemma is all we need to finish the construction of the Gysin morphism of a projective morphism :

Lemma 2.6. Consider a commutative diagram

$$\begin{array}{ccccc} & & P & \xrightarrow{p} & \\ & i & \nearrow & & \\ Y & & & & X \\ & j & \searrow & q & \\ & & Q & & \end{array}$$

with X and Y smooth, i (resp. j) a closed immersion of pure codimension $n+d$ (resp. $m+d$), P (resp. Q) a projective bundle over X of dimension n (resp. m) and p, q the canonical projections.

Then, the following diagram is commutative

$$(2.1) \quad \begin{array}{ccccc} & & M(P)((m)) & \xrightarrow{i^*((n))} & \\ & p^* & \nearrow & & \\ M(X)((n+m)) & & & & M(Y)((n+m+d)). \\ & q^* & \searrow & j^*((m)) & \\ & & M(Q)((n)) & & \end{array}$$

Proof. Considering the diagonal embedding $Y \xrightarrow{(i,j)} P \times_X Q$, we divide diagram (2.1) into three part :

$$\begin{array}{ccccc}
 & & M(P)((m)) & & \\
 & \nearrow p^* & \downarrow p'^* & \searrow i^*((m)) & \\
 M(X)((n+m)) & (1) & M(P \times_X Q) & \xrightarrow{-(i,j)^*} & M(Y)((n+m+d)). \\
 & \searrow q^* & \uparrow q'^* & \nearrow j^*((n)) & \\
 & & M(Q)((n)) & &
 \end{array}$$

The commutativity of part (1) is lemma 2.5. The commutativity of part (2) and that of part (3) are equivalent to the case $X = Q$, $q = 1_X$.

Assume we are in this case. We introduce the following morphisms :

$$\begin{array}{ccccc}
 & & Y & & \\
 & \swarrow \gamma & & \searrow i & \\
 Y \times_X P & \xrightarrow{i'} & P & & \\
 p'' \downarrow & & \downarrow p & & \\
 Y & \xrightarrow{j} & X. & &
 \end{array}$$

We can divide diagram (2.1) into :

$$\begin{array}{ccccc}
 & & M(Y)((n+m+d)) & & \\
 & \nearrow \gamma^*((n+d)) & & \nwarrow i^* & \\
 M(Y)((n+d)) & \xleftarrow{j^*} & M(P) & & \\
 p''^* \uparrow & & \uparrow p^* & & \\
 M(Y)((n+m+d)) & \xleftarrow{i^*((m))} & M(P)((m)). & &
 \end{array}$$

Then commutativity of part (4) is corollary 1.25, and that of part (5) follows from lemma 1.22. Proposition 2.2 allows to conclude. \square

Let $f : Y \rightarrow X$ be a projective morphism between smooth schemes. Following the terminology of [Ful98], 6.6, we say that f has codimension d if it can be factored into a closed immersion $Y \rightarrow P$ of codimension e followed by a smooth projection $P \rightarrow X$ of dimension $e - d$. Indeed, the integer d is uniquely determined (cf *loc.cit.* appendix B.7.6). Using the preceding lemma, we can finally introduce the general definition :

Definition 2.7. Let X, Y be smooth schemes and $f : Y \rightarrow X$ be a projective morphism of codimension d .

We define the Gysin morphism associated to f in $DM_{gm}(k)$

$$f^* : M(X) \rightarrow M(Y)((d))$$

by choosing a factorisation of f into $Y \xrightarrow{i} P \xrightarrow{p} X$ where i is a closed immersion of pure codimension $n + d$ into a projective bundle over X of dimension n and p is the canonical projection, and putting :

$$f^* = \left[M(X)((n)) \xrightarrow{l_n(P)} M(P) \xrightarrow{i^*} M(Y)((n+d)) \right]((-n)).$$

Remark 2.8. With that definition and remark 1.13, we see that the Gysin morphism of a projective morphism f induces the usual pushout on the part of motivic cohomology corresponding to Chow groups.

2.2. Properties.

2.2.1. Functoriality.

Proposition 2.9. *Let X, Y, Z be smooth schemes and $Z \xrightarrow{g} Y \xrightarrow{f} X$ be projective morphism of respective codimension m and n .*

Then, in $DM_{gm}(k)$, we get the equality : $g^ \circ f^* = (fg)^*$.*

Proof. We first choose projective bundles P, Q over X , of respective dimensions s and t , fitting into the following diagram :

$$\begin{array}{ccccc}
 & & Q & & \\
 & \nearrow j & \uparrow p' & \nwarrow q & \\
 & P \times_X Q & & & \\
 & \nearrow i' & \searrow q' & & \\
 Q_Y & & P & & \\
 \nearrow q'' & \nearrow i & \searrow p & & \\
 Z & \xrightarrow{g} & Y & \xrightarrow{f} & X.
 \end{array}$$

The prime exponent of a symbol indicates that the morphism is deduced by base change from the morphism with the same symbol. We then have to prove that the following diagram of $DM_{gm}(k)$ commutes :

$$\begin{array}{ccccc}
 & & M(Q)((t)) & & \\
 & \nearrow q^* & \downarrow p'^* & \nwarrow j^* & \\
 & M(P \times_X Q)((s+t)) & & & \\
 & \nearrow q'^* & \searrow i'^* & & \\
 M(P)((s)) & & M(Q_Y)((n+t)) & & \\
 \nearrow p^* & \nearrow i^* & \searrow q''^* & \nwarrow k^* & \\
 M(X) & & M(Y)((n)) & & M(Z)((n+m)).
 \end{array}$$

The commutativity of part (1) is a corollary of lemma 1.22, that of part (2) is lemma 2.5 and that of part (3) follows from lemma 2.6 and corollary 1.25. \square

2.2.2. Projection formula and excess of intersection. From definition 2.7 and proposition 1.16 we obtain straightforwardly the following :

Proposition 2.10. *Consider a cartesian square of smooth schemes*

$$\begin{array}{ccc}
 T & \xrightarrow{g} & Z \\
 q \downarrow & & \downarrow p \\
 Y & \xrightarrow{f} & X
 \end{array}$$

such that f is a projective morphism of codimension n , and the codimension of g equals that of f .

*Then, the relation $f^*p_* = q_*g^*$ holds in $DM_{gm}(k)$.*

Consider now the situation of a cartesian square of smooth schemes

$$\begin{array}{ccc}
 T & \xrightarrow{g} & Z \\
 q \downarrow & & \downarrow p \\
 Y & \xrightarrow{f} & X
 \end{array}$$

such that f is a projective morphism of codimension n , and denote m the codimension of g . Then $m \leq n$, and we call $e = n - m$ the excess of the above cartesian square.

We attach to the above square a vector bundle ξ of rank e , called the excess bundle. Choose $Y \xrightarrow{i} P \xrightarrow{p} X$ a factorisation of f in a closed immersion of codimension r in a projective bundle over X of dimension s . We put $Q = P \times_X Z$, and denote by $N_T Q$ the normal bundle of the induced closed immersion. Then $N_T Q$ is a sub- X -vector bundle of $N_X P$ and we define $\xi = q^* N_X P / N_T Q$. This definition is independent of the choice of P , as showed in [Ful98], proof of prop. 6.6.

The following proposition is now a straightforward consequence of definition 2.7 and the second case of proposition 1.16 :

Proposition 2.11. *Consider a cartesian square of smooth schemes*

$$\begin{array}{ccc} T & \xrightarrow{g} & Z \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

such that f (resp. g) is a projective morphism of codimension n (resp. m). Let ξ be the associated excess bundle and $e = n - m$ be the rank of ξ .

Then, the relation $f^* p_* = (\mathbf{c}_e(\xi) \boxtimes q_*((m))) \circ g^*$ holds in $DM_{gm}(k)$.

2.2.3. *Compatibility with the Gysin triangle.*

Proposition 2.12. *Consider a cartesian square of smooth schemes*

$$\begin{array}{ccc} T & \xrightarrow{j} & Y \\ g \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array}$$

such that f and g are projective morphisms, i and j are closed immersions. Put $U = X - Z$, $V = Y - T$ and let $h : V \rightarrow U$ be the immersion induced by f . Let n, m, p, q be respectively the relative codimension of i, j, f, g .

Then the following diagram is commutative

$$\begin{array}{ccccccc} M(V)((p)) & \rightarrow & M(Y)((p)) & \xrightarrow{j^*} & M(T)((m+p)) & \xrightarrow{\partial_{Y,T}} & M(V)((p))[1] \\ h^* \uparrow & & f^* \uparrow & & \uparrow g^*((n)) & & \uparrow h^* \\ M(U) & \longrightarrow & M(X) & \xrightarrow{i^*} & M(Z)((n)) & \xrightarrow{\partial_{X,Z}} & M(U)[1] \end{array}$$

where the two lines are the obvious Gysin triangles.

Proof. Use the definition of the Gysin morphism and apply lemma 1.22, theorem 1.24. \square

2.2.4. *Gysin morphisms and transfers in the étale case.* In [Dég05b], 1.1 and 1.2 we introduced another Gysin morphism for a finite equidimensional morphism $f : Y \rightarrow X$. Indeed, we denote by ${}^t f$ the finite correspondence from X to Y obtained by transposing the graph of f . To avoid confusion, we will denote by ${}^t f_* : M(X) \rightarrow M(Y)$ the induced morphism.

Proposition 2.13. *Let X and Y be smooth schemes, and $f : Y \rightarrow X$ be an étale cover.*

Then, $f^* = {}^t f_*$.

Proof. Consider the cartesian square of smooth schemes

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{g} & Y \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{f} & X. \end{array}$$

We first prove that ${}^t f'_* f^* = g^* {}^t f_*$. Choose a factorisation $Y \xrightarrow{i} P \xrightarrow{\pi} X$ of f into a closed immersion in a projective bundle over X followed by the canonical projection. The preceding square then divides into

$$\begin{array}{ccccc} Y \times_X Y & \xrightarrow{j} & P \times_X Y & \xrightarrow{q} & Y \\ f' \downarrow & & f'' \downarrow & & \downarrow f \\ Y & \xrightarrow{i} & P & \xrightarrow{\pi} & X. \end{array}$$

The assertion then follows from the commutativity of the following diagram.

$$\begin{array}{ccccc} M(Y \times_X Y) & \xleftarrow{j^*} & M(P \times_X Y) & \xleftarrow{q^*} & M(Y) \\ {}^t f'_* \uparrow & (1) & {}^t f''_* \uparrow & (2) & \uparrow {}^t f_* \\ M(Y) & \xleftarrow{i^*} & M(P) & \xleftarrow{p^*} & M(X) \end{array}$$

The commutativity of part (1) follows from [Dég05b], prop. 2.5.2 (case 1) and that of part (2) from [Dég05b], prop. 2.2.15 (case 3).

Then, considering the diagonal immersion $Y \xrightarrow{\delta} Y \times_X Y$, it suffices to prove in view of prop. 2.9 that $\delta^* \circ {}^t f'_* = 1$. As Y/X is étale, Y is a connected component of $Y \times_X Y$. Thus, $M(Y)$ is a direct factor of $M(Y \times_X Y)$. Then, it is easy to see that δ^* is the canonical projection on this direct factor, and that ${}^t f'_*$ is the canonical inclusion. \square

2.3. Duality pairing for smooth projective schemes.

2.14. We first recall the abstract definition of duality in monoidal categories. Let \mathcal{C} be a symmetric monoidal category with product \otimes and unit $\mathbf{1}$. An object X of \mathcal{C} is said to have a (*strong*) *dual* if there exists an object X^* of \mathcal{C} and two maps

$$\eta : \mathbf{1} \rightarrow X^* \otimes X, \quad \epsilon : X \otimes X^* \rightarrow \mathbf{1}$$

such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{X \otimes \eta} & X \otimes X^* \otimes X \\ & \searrow 1_X & \downarrow \epsilon \otimes X \\ & & X \end{array} \qquad \begin{array}{ccc} X^* & \xrightarrow{\eta \otimes X^*} & X^* \otimes X \otimes X^* \\ & \searrow 1_{X^*} & \downarrow X^* \otimes \epsilon \\ & & X^* \end{array}$$

For any objects Y and Z of \mathcal{C} , we then have a canonical bijection

$$\mathrm{Hom}_{\mathcal{C}}(Z \otimes X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Z, X^* \otimes Y).$$

In other words, $X^* \otimes Y$ is in this case the internal Hom of the pair (X, Y) for any Y . In particular, such a dual is unique up to a canonical isomorphism. If X^* is a dual of X , then X is a dual of X^* .

Suppose \mathcal{C} is a closed symmetric monoidal triangulated category. Denote by $\underline{\mathrm{Hom}}$ its internal Hom. For any objects X and Y of \mathcal{C} the evaluation map

$$X \otimes \underline{\mathrm{Hom}}(X, \mathbf{1}) \rightarrow \mathbf{1}$$

tensored with the identity of Y defines by adjunction a map

$$\underline{\mathrm{Hom}}(X, \mathbf{1}) \otimes Y \rightarrow \underline{\mathrm{Hom}}(X, Y).$$

The object X has a dual if and only if this map is an isomorphism for all objects Y in \mathcal{C} . Indeed, in this case indeed, $X^* = \underline{\mathrm{Hom}}(X, \mathbf{1})$.

2.15. Let X be a smooth projective k -scheme of pure dimension n , and denote by $p : X \rightarrow \mathrm{Spec}(k)$ the canonical projection, $\delta : X \rightarrow X \times_k X$ the diagonal embedding.

Then we can define morphisms

$$\begin{aligned} \eta : \mathbb{Z} &\xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{\delta_*} M(X)(-n)[-2n] \otimes M(X) \\ \epsilon : M(X) \otimes M(X)(-n)[-2n] &\xrightarrow{\delta^*} M(X) \xrightarrow{p_*} \mathbb{Z}. \end{aligned}$$

One checks easily using the properties of the Gysin morphism these maps turn $M(X)(-n)[-2n]$ into the dual of $M(X)$. We thus have obtained :

Theorem 2.16. *Let X/k be a smooth projective scheme.*

Then the couple of morphisms (η, ϵ) defined above is a duality pairing. Thus $M(X)$ admits a strong dual which is $M(X)(-n)[-2n]$.

Remark 2.17. This fact was proved in [FSV00][chap. 5, th. 4.3.2] using resolution of singularities. Here we obtain a direct proof which does not use this hypothesis. Besides, the proof is purely motivic as we really worked inside the triangulated category of mixed motives.

Note the Gysin morphism $p^* : \mathbb{Z}(n)[2n] \rightarrow M(X)$ defines indeed a homological class η_X in $H_{2n,n}^{\mathcal{M}}(X) = \mathrm{Hom}_{DM_{gm}(k)}(\mathbb{Z}(n)[2n], M(X))$.

The duality above induces an isomorphism

$$H_{\mathcal{M}}^{p,q}(X) \rightarrow H_{p-2n,q-n}^{\mathcal{M}}(X)$$

which is by definition the cap-product by η_X . Thus our duality pairing implies the classical form of Poincaré duality and the class η_X is the fundamental class of X .

3. MOTIVIC CONIVEAU EXACT COUPLE

3.1. Definition.

3.1.1. *Triangulated exact couple.* We introduce a triangulated version of the classical exact couples.

Definition 3.1. Let \mathcal{T} be a triangulated category. objects D and E of \mathcal{T} and homogeneous morphisms between them

$$\begin{array}{ccc} D & \xrightarrow{(1,-1)} & D \\ & \searrow \gamma & \swarrow \beta \\ & E & \end{array} \quad \begin{array}{l} \alpha \\ (0,-1) \\ (-1,1) \end{array}$$

with the bidegrees indicated and such that the above triangle is a distinguished triangle in each bidegree.

Given such a triangulated exact couple, we will usually put $d = \beta \circ \gamma$, homogeneous endomorphism of E of bidegree $(-1, 0)$. We easily get that $d^2 = 0$, thus obtaining a complex

$$\dots \rightarrow E_{p,q} \xrightarrow{d_{p,q}} E_{p-1,q} \rightarrow \dots$$

Let \mathcal{A} be an abelian category. A cohomological functor with values in \mathcal{A} is an additive functor $H : \mathcal{T}^{op} \rightarrow \mathcal{A}$ which sends distinguished triangles to long exact sequences. For p an integer, we simply put $H^p = H \circ [-p]$.

Considering such a cohomological functor, the bigraded objects $H(D)$ and $H(E)$, along with the images of the structural morphisms under H , defines an exact couple in \mathcal{A} in the classical sense (cf [Hu61]). Thus we can associate to this latter exact couple a spectral sequence

$$E_1^{p,q} = H(E_{p,q})$$

with differentials being $H(d_{p,q}) : H(E_{p,q}) \rightarrow H(E_{p-1,q})$.

Definition 3.2. Let \mathcal{T} be a triangulated category, and X an object of \mathcal{T} .

- (1) A tower over X is the data of a sequence $(X^p \rightarrow X)_{p \in \mathbb{Z}}$ of objects over X and a sequence of morphisms over X

$$\dots \rightarrow X^p \xrightarrow{j^p} X^{p+1} \rightarrow \dots$$

- (2) Let X^\bullet be a tower over X . Suppose that for each integer p we are given a distinguished triangle

$$X^p \xrightarrow{j^p} X^{p+1} \xrightarrow{\pi^p} C^p \xrightarrow{\delta^p} X^p[1]$$

where j^p is the structural morphism of the tower X^\bullet .

Then we associate to the tower X^\bullet and the choice of cones C^\bullet a triangulated exact couple

$$D_{p,q} = X^p[-p-q], \quad E_{p,q} = C^p[-p-q]$$

with structural morphisms

$$\alpha_{p,q} = j^p[-p-q], \quad \beta_{p,q} = \pi^p[-p-q], \quad \gamma_{p,q} = \delta^p[-p-q].$$

Let $H : \mathcal{T}^{op} \rightarrow \mathcal{A}$ be a cohomological functor. In the situation of this definition, we thus have a spectral sequence of E_1 -term: $E_1^{p,q} = H^{p+q}(C^p)$.

We consider the case where X^\bullet is bounded and exhaustive *i.e.*

$$X^p = \begin{cases} 0 & \text{if } p \ll 0 \\ X & \text{if } p \gg 0. \end{cases}$$

In this case, the spectral sequence is concentrated in a band with respect to p , thus degenerates. As X^\bullet is exhaustive, we finally get a convergent spectral sequence

$$E_1^{p,q} = H^{p+q}(C^p) \Rightarrow H^{p+q}(X).$$

The filtration on the abutment is then given by the formula

$$Filt^r(H^{p+q}(X)) = \text{Ker}(H^{p+q}(X) \rightarrow H^{p+q}(X^r)).$$

3.1.2. *Definition.* We apply the preceding formalism to the classical coniveau filtration on schemes which we now recall.

Definition 3.3. Let X be a scheme.

A *flag* on X is a decreasing sequence $(Z^p)_{p \in \mathbb{N}}$ of closed subschemes of X such that for all integer $p \geq 0$, Z^p is of codimension greater than p in X . We let $\mathcal{D}(X)$ be the set of flags of X , ordered by termwise inclusion.

We will consider a flag $(Z^p)_{p \in \mathbb{N}}$ has a \mathbb{Z} -sequence by putting $Z^p = X$ for $p < 0$. It is an easy fact that, with the above definition, $\mathcal{D}(X)$ is right filtering.

Recall that a pro-object of a category \mathcal{C} is simply a (*covariant*) functor F from a left filtering category \mathcal{I} to the category \mathcal{C} . Usually, we will denote F by the intuitive notation $\varprojlim_{i \in \mathcal{I}} F_i$.

Definition 3.4. Let X be a scheme. We define the coniveau filtration of X as the sequence $(F^p X)_{p \in \mathbb{Z}}$ of pro-open subschemes of X such that :

$$F^p X = \varprojlim_{Z^* \in \mathcal{D}(X)^{op}} (X - Z^p).$$

We denote by $j^p : F^p X \rightarrow F^{p+1} X$ the canonical pro-open immersion,

$$j^p = \varprojlim_{Z^* \in \mathcal{D}(X)^{op}} (X - Z^p \rightarrow X - Z^{p+1}).$$

Unfortunately, this is a filtration by pro-schemes, and if we apply to it the functor M termwise, we obtain a filtration of $M(X)$ in the category $\text{pro-}DM_{gm}^{eff}(k)$.

This latter category is never triangulated. Nonetheless, the definition of an exact couple obviously still makes sense for a pro-triangulated category. Indeed, we consider the tower of pro-motives above $M(X)$

$$\dots \rightarrow M(F^p X) \xrightarrow{j_*^p} M(F^{p+1} X) \rightarrow \dots$$

We define the following canonical pro-cone

$$Gr_M^p(X) = \varprojlim_{Z^* \in \mathcal{D}(X)^{op}} M(X - Z^p / X - Z^{p-1}).$$

using definition 1.2 and its functoriality. We thus obtain pro-distinguished triangles:

$$M(F^p X) \xrightarrow{j_*^p} M(F^{p+1} X) \xrightarrow{\pi^p} Gr_M^p(X) \xrightarrow{\delta^p} M(F^p X)[1].$$

Definition 3.5. Consider the above notations. We define the motivic coniveau exact couple associated to X in $\text{pro-}DM_{gm}^{eff}(k)$ as

$$D_{p,q} = M(F^p X)[-p-q], \quad E_{p,q} = Gr_M^p(X)[-p-q],$$

with structural morphisms

$$\alpha_{p,q} = j^p[-p-q], \quad \beta_{p,q} = \pi^p[-p-q], \quad \gamma_{p,q} = \delta^p[-p-q].$$

3.2. Computations.

3.2.1. *Recall and complement on generic motives.* We call simply *function field* any finite type extension field E/k . A *model* of the function field E will be a connected smooth scheme X/k with a given k -isomorphism between the function field of X and E . Recall the following definition from [Dég05b, 3.3.1] :

Definition 3.6. Consider a function field E/k and an integer $n \in \mathbb{Z}$. We define the *generic motive of E with weight n* as the following pro-object of $DM_{gm}(k)$:

$$M(E) := \varprojlim_{A \subset E, \text{Spec}(A) \text{ model of } E/k} M(\text{Spec}(A))(n)[n].$$

We denote by $DM_{gm}^{(0)}(k)$ the full subcategory of $\text{pro-}DM_{gm}(k)$ made by the generic motives.

Of course, given a function field E/k with model X/k , the pro-object $M(E)$ is canonically isomorphic to the pro-motive made by the motives of non empty open subschemes of X .

3.7. The interest of generic motives lies in their functoriality which we now recall :

- (1) Given any extension of function fields $\varphi : E \rightarrow L$, we get a morphism $\varphi^* : M(L) \rightarrow M(E)$ (by covariant functoriality of motives).
- (2) Consider a finite extension of function field $\varphi : E \rightarrow L$. One can find respective models X and Y of E and L together with a finite morphism of schemes $f : Y \rightarrow X$ which induces on function fields the morphism φ through the structural isomorphisms.

For any open subscheme $U \subset X$, we put $Y_U = Y \times_X U$ and let $f_U : Y_U \rightarrow U$ be the morphism induced by f . It is finite and surjective. In particular, its graph seen as a cycle in $U \times Y_U$ defines a finite correspondence from U to Y_U , denoted by ${}^t f_U$ and called the transpose of f_U . We define the *norm morphism* $\varphi_* : M(E) \rightarrow M(L)$ as the well defined pro-morphism (see [Dég05b, 5.2.9])

$$\varprojlim_{U \subset X} \left(M(U) \xrightarrow{({}^t f_U)_*} M(Y_U) \right)$$

through the structural isomorphisms of the models X and Y .

- (3) Consider a function field E and a unit $x \in E^\times$. Given a smooth sub- k -algebra $A \subset E$ such that $x \in A$, we get a morphism $f_A : \text{Spec}(A) \rightarrow \mathbb{G}_m$. Recall the canonical decomposition $M(\mathbb{G}_m) = \mathbb{Z} \oplus \mathbb{Z}(1)[1]$ and consider the associated projection $M(\mathbb{G}_m) \xrightarrow{\pi} \mathbb{Z}(1)[1]$. We associate to the unit x the morphism $\gamma_x : M(E) \rightarrow M(E)(1)[1]$ defined as

$$\varprojlim_{x \in A \subset E} \left(M(\text{Spec}(A)) \xrightarrow{f_{A*}} M(\mathbb{G}_m) \xrightarrow{\pi} \mathbb{Z}(1)[1] \right).$$

One can prove moreover that if $x \neq 1$, $\gamma_x \circ \gamma_{1-x} = 0$ and $\gamma_{1-x} \circ \gamma_x = 0$ so that any element $\sigma \in K_n^M(E)$ of Milnor K-theory defines a morphism $\gamma_\sigma : M(E) \rightarrow M(E)(n)[n]$ (see also [Dég05b, 5.3.5]).

- (4) Let E be a function field and v a discrete valuation on E with ring of integers \mathcal{O}_v essentially of finite type over k . Let $\kappa(v)$ be the residue field of v .

As k is perfect, there exists a connected smooth scheme X with a codimension 1 point x such that $\mathcal{O}_{X,x}$ is isomorphic to \mathcal{O}_v . This implies X is a model of E/k . Moreover, reducing X , one can assume the closure Z of x in X is smooth so that it becomes a model of $\kappa(v)$.

For an open subscheme $U \subset X$ containing x , we put $Z_U = Z \times_X U$. We define

the *residue morphism* $\partial_v : M(\kappa(v))(1)[1] \rightarrow M(E)$ associated to (E, v) as the pro-morphism

$$\varprojlim_{x \in U \subset X} (M(Z_U)(1)[1] \xrightarrow{\partial_{U, Z_U}} M(U - Z_U)).$$

The fact this pro-morphism is well defined evidently relies on the transversal case of 1.16 (see also [Dég05b, 5.4.6]).

Remark 3.8. All these morphisms satisfy a set of relations which is most optimally described in the axioms of a cycle premodule by M. Rost (cf [Ros96, (1.1)]). We refer the reader to [Dég05b, 5.1.1] for a precise statement.

3.9. Consider again the situation and notations of the point (2) in paragraph 3.7. With the Gysin morphism we have introduced before, one can give another definition for the norm morphism of generic motives.

Indeed, for any open subscheme U of X , the morphism $f_U : Y_U \rightarrow U$ is finite of relative dimension 0 and thus induces a Gysin morphism $f_U^* : M(U) \rightarrow M(Y_U)$. Using proposition 2.10, these morphisms are natural with respect to U . Thus, we get a morphism of pro-objects

$$\varprojlim_{U \subset X} (M(U) \xrightarrow{f_U^*} M(Y_U)).$$

which induces through the structural isomorphisms of the models X and Y a morphism $\varphi'_* : M(E) \rightarrow M(L)$.

Lemma 3.10. *Consider the notations above. Then, $\varphi'_* = \varphi_*$.*

Proof. By functoriality, we can restrict the proof to the cases where L/E is separable or L/E is purely inseparable.

In the first case, we can choose a model $f : Y \rightarrow X$ of φ which is étale. Then the lemma follows from proposition 2.13.

In the second case, we can suppose that $L = E[\sqrt[q]{a}]$ for $a \in E$. Let $A \subset E$ be a sub- k -algebra containing a such that $X = \text{Spec}(A)$ is a smooth scheme. Let $B = A[t]/(t^q - a)$. Then $Y = \text{Spec}(B)$ is again a smooth scheme and the canonical morphism $f : Y \rightarrow X$ is a model of L/E . We consider its canonical factorisation $Y \xrightarrow{i} \mathbb{P}_X^1 \xrightarrow{p} X$ corresponding to the parameter t and the following diagram of cartesian squares

$$\begin{array}{ccccc} Y \times_X Y & \xrightarrow{j} & \mathbb{P}_Y^1 & \xrightarrow{q} & Y \\ \downarrow & & \downarrow f' & & \downarrow f \\ Y & \xrightarrow{i} & \mathbb{P}_X^1 & \xrightarrow{p} & X. \end{array}$$

The scheme $Y \times_X Y$ is non reduced and its reduction is Y . Moreover, the canonical immersion $Y \rightarrow Y \times_X Y$ is an exact thickening of order q in Y (cf def. 2.4.7 of [Dég05b]). Thus, the following diagram is commutative :

$$\begin{array}{ccccc} M(Y) & \xleftarrow{j^*} & M(\mathbb{P}_Y^1) & \xleftarrow{q^*} & M(Y) \\ \parallel & (1) & \uparrow {}^t f'_* & (2) & \uparrow {}^t f_* \\ M(Y) & \xleftarrow{i^*} & M(\mathbb{P}_X^1) & \xleftarrow{p^*} & M(X). \end{array}$$

Indeed, part (2) (resp. (1)) is commutative by [Dég05b, 2.2.15] (resp. [Dég05b, 2.5.2: (2)]). Thus $f^* = {}^t f_*$ and an easy localization argument allows to conclude. \square

3.2.2. The graded terms. For a scheme X , we let $X^{(p)}$ denote the set of points of X of codimension p . If x is a point of X , $\kappa(x)$ denotes its residue field. The symbol " \prod " denotes the product in the category of pro-motives.

Lemma 3.11. *Let X be a smooth scheme.*

Then, for all integer $p \geq 0$, the purity isomorphism 1.9 induces a canonical isomorphism

$$Gr_M^p(X) = \prod_{x \in X^{(p)}} M(\kappa(x))(p)[2p].$$

Proof. Let \mathcal{I}_p be the set of pairs (Z, Z') such that Z is a reduced closed subset of X of codimension p , Z' is a closed subset of Z containing its singular locus. Then

$$Gr_M^p(X) = \varprojlim_{(Z, Z') \in \mathcal{I}_p} M(X - Z'/X - Z).$$

For any element (Z, Z') of \mathcal{I}_p , under the purity isomorphism, we get:

$$M(X - Z'/X - Z) = M(Z - Z')(p)[2p].$$

For any point x of X , we let $Z(x)$ be the reduced closure of x in X and $\mathcal{F}(x)$ be the set of closed subschemes Z' of $Z(x)$ containing the singular locus $Z(x)_{\text{sing}}$. By additivity of motives and the purity isomorphism just recalled, we finally get

$$Gr_M^p(X) = \prod_{x \in X^{(p)}} \varprojlim_{Z' \in \mathcal{F}(x)} M(Z(x) - Z').$$

This implies the lemma, because $Z(x) - Z(x)_{\text{sing}}$ is a model of $\kappa(x)$. \square

3.2.3. The differentials.

3.12. Let X be a scheme essentially of finite type³ over k . Consider a point x of codimension p in X and y a specialization of x , of codimension $p + 1$ in X . Let Z be the integral closure of x in X and $\tilde{Z} \xrightarrow{f} Z$ be its normalisation. Each point $z \in f^{-1}(y)$ corresponds to a discrete valuation v_z of $\kappa(x)$ with residue field $\kappa(z)$. We denote by $\varphi_z : \kappa(y) \rightarrow \kappa(z)$ the morphism induced by f . Then, we define the morphism of generic motives

$$\partial_y^x : M(\kappa(y))(1)[1] \xrightarrow{\varphi_{z*}} M(\kappa(z))(1)[1] \xrightarrow{\partial_{v_z}} M(\kappa(x))$$

using the notations of 3.7.

Proposition 3.13. *Let X be a smooth scheme.*

Then, for all integer $p \geq 0$, the differential

$$d^p : Gr_M^{p+1}(X) \rightarrow Gr_M^p(X)[1]$$

of the exact couple of definition 3.5 is equal, through the isomorphism of 3.11, to the well defined morphism

$$\prod_{x, y} \partial_y^x : \prod_{y \in X^{(p+1)}} M(\kappa(y))(p+1)[2p+2] \rightarrow \prod_{x \in X^{(p)}} M(\kappa(x))(p)[2p].$$

³ For the purpose of the next proposition, we need only the case when X is smooth but the general case treated here will be used latter.

Proof. Fix a point x of codimension p in X . We have to identify the composition

$$Gr_M^{p+1}(X) \xrightarrow{d^p} Gr_M^p(X)[1] \rightarrow M(\kappa(x))(p)[2p+1]$$

where the second morphism is defined through the isomorphism of lemma 3.11. For that purpose, we can always subtract a codimension $p+2$ closed subset of X . Let Z be the reduced closure of x in X . The previous morphism is the pseudo-projective limit of the morphisms

$$M(X - W/X - Y) \rightarrow M(X - Y)[1] \rightarrow M(X - Y/X - Z)[1],$$

for varying codimension 1 closed subsets $Y \subset Z$ and $W \subset Y$.

Fix $W \subset Y \subset X$ among these closed subsets aiming to compute the pseudo-projective limit mentionned above. Subtracting $W \cup Y_{sing}$ to X , we can assume that W is empty and Y is smooth. Through the purity isomorphisms, the above morphism then takes the form

$$M(Y)(p+1)[2p+2] \xrightarrow{\partial_{X,Y}} M(X - Y)[1] \xrightarrow{i^*[1]} M(Z - Y)(p)[2p+1]$$

where $i : (Z - Y) \rightarrow (X - Y)$ is the canonical closed immersion.

Let \tilde{Z} be the normalization of Z , and $f : \tilde{Z} \rightarrow Z$ the canonical projection. The singular locus \tilde{Z}_{sing} of \tilde{Z} is everywhere of codimension greater than 2 in \tilde{Z} . Thus, $f(\tilde{Z}_{sing})$ is everywhere of codimension greater than $p+2$ in X , and we can assume by reducing X again that \tilde{Z} is smooth.

Let $\tilde{Y} = f^{-1}(Y)$ with its reduced structure of closed subscheme of \tilde{Z} . Reducing once again X , we can assume \tilde{Y} is smooth. We let $g : \tilde{Y} \rightarrow Y$ and $h : (\tilde{Z} - \tilde{Y}) \rightarrow (Z - Y)$ be the morphism induced by f . Then, according to 2.12 and the first commutative square of theorem 1.24, the following diagram is commutative

$$\begin{array}{ccccc} M(Y)((p+1)) & \xrightarrow{\partial_{X,Y}} & M(X - Y)[1] & \xrightarrow{i^*} & M(Z - Y)((p))[1] \\ g^* \downarrow & & & & \downarrow h^* \\ M(\tilde{Y})((p+1)) & \xrightarrow{\partial_{\tilde{Y}, \tilde{Z}}} & M(\tilde{Z} - \tilde{Y})((p))[1] & & \end{array}$$

This concludes as the morphism h is birational and thus h^* induces the identity on $M(\kappa(x))$ when passing to the limit over Y . \square

4. COHOMOLOGICAL REALIZATION

We fix a Grothendieck abelian category \mathcal{A} and consider a cohomological functor

$$H : DM_{gm}(k)^{op} \rightarrow \mathcal{A},$$

simply called a *realization functor*.

To the realization functor H , we can associate a twisted cohomological theory such that for a smooth scheme X and a pair of integers $(n, i) \in \mathbb{Z}^2$,

$$H^n(X, i) = H(M(X)(-i)[-n]).$$

By the very definition, this functor is contravariant, not only with respect to morphisms of smooth schemes but also for finite correspondences. According to the construction of definition 2.7, it is covariant with respect to projective morphisms.

4.1. The coniveau spectral sequence. The functor H admits an obvious extension to pro-objects $\bar{H} : \text{pro-}DM_{gm}(k)^{op} \rightarrow \mathcal{A}$ which sends pro-distinguished triangles to long exact sequences since right filtering colimits are exact in \mathcal{A} . In particular, for any function fields E/k , we define

$$\bar{H}^r(E, n) = \varinjlim_{A \subset E} H^r(\text{Spec}(A), n)$$

where the limit is taken over the models of E/k .

Fix an integer $n \in \mathbb{Z}$. We apply the functor $\bar{H}(?(n))$ to the pro-exact couple of 3.5. We then obtain a converging spectral sequence which, according to lemma 3.11, has the form:

$$(4.1) \quad E_1^{p,q}(X, n) = \bigoplus_{x \in X^{(p)}} \bar{H}^{q-p}(\kappa(x), n-p) \Rightarrow H^{p+q}(X, n).$$

This is the coniveau spectral sequence of X with coefficients in H .

Remark 4.1. (Bloch-Ogus theory) The filtration on $H^*(X, n)$ which appears on the abutment of the spectral sequence (4.1) is the filtration which appears originally⁴ in [Gro69] and [Gro68, 1.10],

$$N^r H^*(X, n) = \text{Ker} \left(H^*(X, n) \rightarrow \bar{H}(M^{(r)}(X)(n)[*]) \right),$$

formed by cohomology classes which vanish on an open subset with complementary of (at least) codimension r .

One can relate this spectral sequence to the one introduced in [BO74], 3.11. Indeed, without referring to the duality for the cohomological theory H^* , we can obviously extend H^* to a cohomology theory with support using relative motives. This is all what we need to define the spectral sequence 3.11 of *loc. cit.* Then the latter spectral sequence coincides with spectral sequence (4.1).

4.2. Cycle modules. Cycle modules have been introduced by M. Rost in [Ros96] as a notion of "coefficient systems" suitable to define "localization complexes for varieties". We recall below this theory in a way suitable for our needs.

4.2. The first step in Rost's theory is the notion of a *cycle premodule*. Basically, it is a covariant functor from the category of function fields to the category of graded abelian groups satisfying an enriched functoriality exactly analog to that of Milnor K-theory K_*^M . In our context, we will define⁵ a cycle premodule as a functor

$$K : DM_{gm}^{(0)}(k)^{op} \rightarrow \mathcal{A}.$$

Usually, we put $K(M(E)(-n)[-n]) = K_n(E)$ so that K becomes a graded functor on function fields. In view of the description of the functoriality of generic motives recalled in 3.7, such a functor satisfy the following functoriality :

- (1) For any extension of function fields, $\varphi : E \rightarrow L$, a *corestriction* $\varphi_* : K_*(E) \rightarrow K_*(L)$ of degree 0.
- (2) For any finite extension of function fields, $\varphi : E \rightarrow L$, a *norm* $\varphi^* : K_*(L) \rightarrow K_*(E)$ of degree 0, also denoted by $N_{L/E}$.

⁴ In [Gro69], the filtration is called "filtration arithmétique" and in [Gro68], "filtration par le type dimensionnel". One can also find in the latter article the root of the actual terminology, filtration by niveau, which was definitively adopted after the fundamental work of [BO74].

⁵Indeed, when \mathcal{A} is the category of abelian groups, it is proven in [Dég05b, th. 5.1.1] that such a functor defines a cycle premodule in the sense of M. Rost.

- (3) For any function field E , $K_*(E)$ admits a $K_*^M(E)$ -graded module structure.
- (4) For any valued function field (E, v) with ring of integers essentially of finite type over k and residue field $\kappa(v)$, a *residue* $\partial_v : K_*(E) \rightarrow K_*(\kappa(v))$ of degree -1 .

Definition 4.3. Consider again a realization functor H . For any pair of integers (q, n) , we associate to H a cycle module $\hat{H}_*^{q,n}$ as the restriction of the functor $\bar{H}^q(., n)$ to the category $DM_{gm}^{(0)}(k)$.

Concretely, $\hat{H}_{-p}^{q,n}(E) = \bar{H}^{q-p}(E, n-p)$. Remark that,

$$(4.2) \quad \forall a \in \mathbb{Z}, \hat{H}_*^{q-a, n-a} = \hat{H}_{*+a}^{q,n}$$

and this is an equality (up to the decalage) of cycle modules. In our notation, the choice of the grading is somewhat redundant but it will be convenient for our needs.

4.4. To the notion of a cycle premodule K is attached by Rost axioms which allow to write a complex with coefficients in K (cf [Ros96, (2.1)]). We recall these axioms to the reader using the morphisms introduced in 3.12. We say that a cycle premodule K is a *cycle module* if the following two conditions are fulfilled :

- (FD) Let X be a normal scheme essentially of finite type over k , η its generic point and E its functions field, and consider an element $\rho \in K_i(E)$. Then $K(\partial_x^\eta)(\rho) = 0$ for all but finitely many points x of codimension 1 in X .
- (C) Let X be an integral local scheme essentially of finite type over k and of dimension 2. Let η (resp. s) be its generic (resp. closed) point, and E (resp. κ) be its function (resp. residue) field. Then, for any integer $n \in \mathbb{Z}$, the morphism

$$\sum_{x \in X^{(1)}} K_{n-1}(\partial_s^x) \circ K_n(\partial_x^\eta) : K_n(E) \rightarrow K_{n-2}(\kappa),$$

well defined under (FD), is zero.

When these conditions are fulfilled, for any scheme X essentially of finite type over k , we define according to [Ros96, (3.2)] a graded complex of *cycles with coefficients in K* whose i -th graded⁶ p -cochains are

$$C^p(X; K)_i = \bigoplus_{x \in X^{(p)}} K_{i-p}(\kappa(x))$$

and with p -th differentials equal to the well defined morphism

$$d^p = \sum_{x \in X^{(p)}, y \in \overline{\{x\}}^{(1)}} K(\partial_x^y).$$

The cohomology groups of this complex are called the *Chow groups with coefficients in K* and denoted by $A^*(X; K)$ in [Ros96]. They are graded according to the graduation on $C^*(X; K)$.

As a corollary of 3.13, we obtain the following result :

Proposition 4.5. *Consider the previous notations.*

⁶ This graduation follows the convention of [Ros96], part 5 except for the notation. The notation of Rost $C^p(X; K, i)$ would introduce a confusion with twists.

- (i) For any smooth scheme X , and any couple of integers (n, q) , there is a canonical isomorphism of complex

$$E_1^{*,q}(X, n) = C^*(X; \hat{H}^{q,n})_0,$$

where the left hand side is the E_1 -term of (4.1).

- (ii) For any integer $q \in \mathbb{Z}$, the cycle premodule $\hat{H}^{q,n}$ is a cycle module.

Proof. The point (i) follows from 3.11 for the construction of the isomorphism and from 3.13 for the identification of the differentials.

We prove point (ii), axiom (FD). Consider a normal scheme X essentially of finite type over k . We can assume it is affine of finite type. Then, there exists a closed immersion $X \xrightarrow{i} \mathbb{A}_k^r$ for an integer $r \geq 0$. From point (i), for any integer $a \in \mathbb{Z}$, the sequence $C^*(\mathbb{A}_k^r; \hat{H}^{q-a, n-a})_0$ is a well defined complex, equal to $C^*(\mathbb{A}_k^r; \hat{H}^{q,n})_a$ according to (4.2). Thus, axiom (FD) for the cycle premodule $\hat{H}^{q,n}$ follows from the fact

$$\hat{H}_a^{q,n}(E) \subset C^r(\mathbb{A}_k^r; \hat{H}^{q,n})_a$$

and the definition of the differentials given above.

For axiom (C), we consider an integral local scheme X essentially of finite type over k and of dimension 2. We have to prove $C^*(X; \hat{H}^{q,n})$ is a complex – the differentials are well defined according to (FD). For that, we can assume X is affine of finite type over k . Then, there exists a closed immersion $X \rightarrow \mathbb{A}_k^r$. From the definition given above, for any integer $a \in \mathbb{Z}$, we obtain a monomorphism

$$C^p(X; \hat{H}^{q,n})_a \rightarrow C^p(\mathbb{A}_k^r; \hat{H}^{q,n})_a = C^p(\mathbb{A}_k^r; \hat{H}^{q-a, n-a})_0$$

which is compatible with differentials. The conclusion then follows from point (i). \square

Remark 4.6. This corollary gives an alternative proof of the main theorem [Dég05b, 6.2.1] concerning the second affirmation.

Corollary 4.7. *Using the notations of the previous proposition, the E_2 -term of the coniveau spectral sequence (4.1) are :*

$$E_2^{p,q}(X, n) = A^p(X; \hat{H}^{q,n})_0 \Rightarrow H^{p+q}(X, n).$$

Moreover, for any integers (q, n) and any smooth proper scheme X , the term $E_2^{0,q}(X, n)$ is a birational invariant of X .

The second assertion follows from [Ros96, 12.10].

Example 4.8. Consider the functor $H_{\mathcal{M}} = \text{Hom}_{DM_{gm}(k)}(., \mathbb{Z})$, corresponding to motivic cohomology. In this case, following [SV00, 3.2, 3.4], for any field E ,

$$H_{\mathcal{M}}^q(E; \mathbb{Z}(p)) = \begin{cases} 0 & \text{if } q > p \text{ or } p < 0 \\ K_p^M(E) & \text{if } q = p \end{cases}$$

In particular, from definition 4.3, $\hat{H}_{\mathcal{M}}^{n,n} = K_{*+n}^M$. In fact, this is an isomorphism of cycle modules. For the norm, this is *loc. cit.* 3.4.1. For the residue, we easily reduce (using for example [Ros96, formula (R3f)]) to prove that for any valued function field (E, v) with uniformizing parameter π , $\partial_v(\pi) = 1$ for the cycle module $\hat{H}_{\mathcal{M}}^{n,n}$. This now follows from [Dég05b, 2.6.5] as for any morphism of smooth connected schemes $f : Y \rightarrow X$, the pullback $f^* : H_{\mathcal{M}}^0(X; \mathbb{Z}) \rightarrow H_{\mathcal{M}}^0(Y; \mathbb{Z})$ is the identity of \mathbb{Z} .

As remarked by Voevodsky at the very beginning of his theory, the vanishing mentionned above implies that the coniveau spectral sequence for $H_{\mathcal{M}}$ satisfies $E_1^{p,q}(X, n) = 0$ if $p > n$ or $q > n$. This immediately gives that the edge morphisms of this spectral sequence induces an isomorphism $A^n(X; \hat{H}^{n,n})_0 \rightarrow H_{\mathcal{M}}^{2n}(X; \mathbb{Z}(n))$. The left hand side is $A^n(X; K_*^M)_n$ and an easy verification shows this group is $CH^n(X)$.

4.9. In the sequel, we will need the following functoriality of the Chow group of cycles with coefficients in a cycle module K :

- $A^*(.; K)$ is contravariant for flat morphisms ([Ros96, (3.5)]).
- $A^*(.; K)$ is covariant for proper morphisms ([Ros96, (3.4)]).
- For any smooth scheme X , $A^*(X; K)$ is a graded module over $CH^*(X)$ ([Dég06, 5.7 and 5.12]).

Note that any morphism of cycle modules gives a transformation on the corresponding Chow group which is compatible with the structures listed above. Moreover, identifying $A^p(.; K_*^M)_p$ with $CH^p(.,)$, as already mentionned in the preceding example, the structures above corresponds to the usual structures on the Chow group. Finally, let us recall that these structural maps are defined at the level of the underlying complexes.

In [BO74], the authors expressed the E_2 -term of the coniveau spectral sequence as the Zariski cohomology of a well defined sheaf. We get the same result in our setting. Recall from [FSV00], chap. 5 that a sheaf with transfers is an additive functor $F : (\mathcal{S}m_k^{cor})^{op} \rightarrow \mathcal{A}b$ which induces a Nisnevich sheaves on the category of smooth schemes. Let $\mathcal{H}^q(n)$ be the presheaf on the category of smooth schemes such that $\Gamma(X; \mathcal{H}^q(n)) = A^0(X; \hat{H}^{q,n})_0$. This group is called the *n-th twisted unramified cohomology* of X with coefficients in H .

Proposition 4.10. *Consider the notations above.*

- (1) *The presheaf $\mathcal{H}^q(n)$ is a homotopy invariant Nisnevich sheaf. It has a canonical structure of a sheaf with transfers.*
- (2) *There are natural isomorphisms*

$$A^p(X; \hat{H}^{q,n})_0 = H_{\text{Zar}}^p(X; \mathcal{H}^q(n)).$$

Proof. The first assertion follows from [Dég06, 6.9] and the second one from [Ros96, (2.6)]. \square

Finally, we have obtained the following form for the coniveau spectral sequence

$$(4.3) \quad E_2^{p,q}(X, n) = H_{\text{Zar}}^p(X; \mathcal{H}^q(n)) \Rightarrow H^{p+q}(X, n).$$

Remark 4.11. By definition, the presheaf $\mathcal{H}^q(?, n)$ is a presheaf with transfers. For any smooth scheme X , there is a canonical map

$$H^q(X, n) \rightarrow \Gamma(X; \mathcal{H}^q(n)).$$

One can check this map is compatible with transfers so that we get a morphism of presheaves with transfers

$$H^q(?, n) \rightarrow \mathcal{H}^q(n).$$

By definition, the fibre of this map on any function field is an isomorphism. Thus, it follows from one of the main point of Voevodsky's theory (cf [Dég05a, 4.4.8]) that $\mathcal{H}^q(n)$ is the Zariski sheaf associated to $H^q(?, n)$. Thus we recover in our setting the form of the coniveau spectral sequence obtained in [BO74].

4.3. Algebraic equivalence. In this section, we assume \mathcal{A} is the category of K -vector spaces for a given field K . We assume furthermore the following conditions on the realization functor H :

- (*Vanishing*) For any function field E and any couple of negative integers (q, n) , $\bar{H}^q(E, n) = 0$.
- (*Rigidity*) (i) $H^0(\text{Spec}(k)) = K$.
- (ii) For any function field E , the corestriction $\bar{H}^0(k) \rightarrow \bar{H}^0(E)$ is an isomorphism.

The element $1 \in H^0(\text{Spec}(k)) = H(\mathbb{Z})$ determines a natural transformation $H_{\mathcal{M}} = \text{Hom}_{DM_{gm}(k)}(., \mathbb{Z}) \xrightarrow{\sigma} H$. In particular, we get a cycle class

$$CH^p(X)_K \xrightarrow{\sigma_X^p} H^{2p}(X, p).$$

We put $\mathcal{K}_H^p(X) = \text{Ker}(\sigma_X^p)$. We denote by $\mathcal{K}_{alg}^p(X)_K$ the subgroup of $CH^p(X)_K$ made of K -cycles algebraically equivalent to 0. $A^p(X)_K = CH^p(X)_K / \mathcal{K}_{alg}^p(X)_K$.

According to example 4.8, the morphism σ induces a morphism of cycle modules $K_{*+a}^M \rightarrow \hat{H}^{a,a}$ which correspond to cohomological symbols

$$K_a^M(E) \rightarrow \bar{H}^a(E, a),$$

compatible with corestriction, norm, residues and the action of $K_*^M(E)$. The natural transformation σ induces a morphism of the coniveau spectral sequences which on the E_2 -terms induces a morphism

$$A^p(X; \hat{H}_{\mathcal{M}}^{q,n})_0 \rightarrow A^p(X; \hat{H}^{q,n})_0.$$

Taking $p = q = n$ and applying example 4.8, proposition 4.10, we get a canonical map

$$\tilde{\sigma}_X^p : CH^p(X)_K \rightarrow H_{\text{Zar}}^p(X; \mathcal{H}^p(p)).$$

According to the rigidity property above, the edge morphisms of the coniveau spectral sequence induce a morphism

$$H_{\text{Zar}}^p(X; \mathcal{H}^p(p)) \xrightarrow{\Phi_X^p} H^{2p}(X, p)$$

and by construction $\sigma_X^p = \Phi_X^p \circ \tilde{\sigma}_X^p$.

The following proposition is a generalization of a result of Bloch-Ogus (cf [BO74, (7.4)]).

Proposition 4.12. *Consider the preceding hypothesis and notations. Assume (*Vanishing*) and (*Rigidity*).*

Then the following conditions are equivalent :

- (i) *For any smooth proper scheme X , $\mathcal{K}_H^1(X) = \mathcal{K}_{alg}^1(X)_K$.*
- (ii) *For any smooth proper scheme X , the map $\tilde{\sigma}_X^p$ induces an isomorphism*

$$A^p(X)_K \rightarrow H_{\text{Zar}}^p(X; \mathcal{H}^p(p)).$$

Proof. Note for $p = 0$, condition (ii) follows from (Rigidity). Moreover, this assumption implies that for any function field E , the symbol map $K_0^M(E) \rightarrow \hat{H}^{p,p}(E)$ are isomorphisms. It readily implies the map σ induces an isomorphism on the E_1 -terms $E_1^{p,p}(X, p)$ of the coniveau spectral sequences. Hence $\tilde{\sigma}_X$ is surjective in any case. We put $\hat{\mathcal{K}}^p(X) = \text{Ker}(\tilde{\sigma}_X^p)$. Then condition (ii) is equivalent to the equality $\hat{\mathcal{K}}^p(X) = \mathcal{K}_{alg}^p(X)$.

The assumption (Vanishing) implies that the coniveau spectral sequence for H degenerates at the term $E_2^{1,1}(X, 1)$. Thus, Φ_X^1 is a monomorphism. In particular, condition (ii) for $p = 1$ is equivalent to condition (i).

To finish the proof, we assume (i) and prove $\mathcal{K}_{alg}^p(X) = \hat{\mathcal{K}}^p(X)$ for any $p > 1$.

For the inclusion $\mathcal{K}_{alg}^p(X) \subset \hat{\mathcal{K}}^p(X)$, we consider $\alpha, \beta \in CH^p(X)$ such that α is algebraically equivalent to β . This means there exists a smooth proper connected curve C , points $x_0, x_1 \in C(k)$, and a cycle γ in $CH^p(X \times C)$ such that $f_*(g^*(x_0) \cdot \gamma) = \alpha$, $f_*(g^*(x_1) \cdot \gamma) = \beta$ where $f : X \times C \rightarrow X$ and $g : X \times C \rightarrow X$ are the canonical projections. Using the functoriality described in paragraph 4.9 applied to the morphism of cycle modules $K_*^M \rightarrow \hat{H}^{0,0}$, we get a commutative diagram

$$\begin{array}{ccccccc} A^1(C; K_*^M) & \xrightarrow{q^*} & A^1(C \times X; K_*^M) & \xrightarrow{\gamma} & A^{p+1}(C \times X; K_*^M) & \xrightarrow{f_*} & A^p(X; K_*^M) \\ (1) \downarrow & & \downarrow & & \downarrow & & \downarrow (2) \\ A^1(C; \hat{H}^{0,0}) & \xrightarrow{q^*} & A^1(C \times X; \hat{H}^{0,0}) & \xrightarrow{\gamma} & A^{p+1}(C \times X; \hat{H}^{0,0}) & \xrightarrow{f_*} & A^p(X; \hat{H}^{0,0}) \end{array}$$

Recall the identifications:

$A^p(X; K_*^M)_p = CH^p(X)$ and $A^p(X; \hat{H}^{0,0})_p = A^p(X; \hat{H}^{p,p})_0 = H_{Zar}^p(X; \mathcal{H}^p(p))$. According to these ones, the first (resp. p -th) graded piece of the map (1) (resp. (2)) is by definition the morphism $\tilde{\sigma}_X^1$ (resp. $\tilde{\sigma}_X^p$). In particular, we are reduced to prove that $x_0 - x_1$ belongs to $\hat{\mathcal{K}}^1(C)$. This case is already treated above.

We prove $\hat{\mathcal{K}}^p(X) \subset \mathcal{K}_{alg}^p(X)$. Recall that $A^p(X; \hat{H}^{p,p})_0$ is the cokernel of the differential

$$C^{p-1}(X; \hat{H}^{p,p})_0 \xrightarrow{d_X^{p-1}} C^p(X; \hat{H}^{p,p})_0 = Z^p(X)_K.$$

We have to prove that the image of this map are cycles algebraically equivalent to zero. Consider a point $y \in X^{(p-1)}$ with residue field E and $\rho \in \bar{H}^{1,1}(E)$. We consider the immersion $Y \xrightarrow{i} X$ of the reduced closure of y in X and consider an alteration $Y' \xrightarrow{f} Y$ such that Y' is smooth over k using De Jong's theorem. Let $\varphi : E \rightarrow L$ be the extension of function fields associated with f . According to the basic functoriality of cycle modules 4.9, we obtain a commutative diagram

$$\begin{array}{ccccccc} \bar{H}^{1,1}(L) & \xlongequal{\quad} & C^0(Y'; \hat{H}^{1,1})_0 & \xrightarrow{d_{Y'}^1} & C^1(Y'; \hat{H}^{1,1})_0 & \xlongequal{\quad} & Z^1(Y') \\ N_{L/E} \downarrow & & \downarrow & & \downarrow & & \downarrow f_* \\ \bar{H}^{1,1}(E) & \xlongequal{\quad} & C^0(Y; \hat{H}^{1,1})_0 & \xrightarrow{d_Y^1} & C^1(Y; \hat{H}^{1,1})_0 & \xlongequal{\quad} & Z^1(Y) \\ & & \downarrow \wr & & \downarrow & & \downarrow i_* \\ & & C^{p-1}(X; \hat{H}^{p,p})_0 & \xrightarrow{d_X^{p-1}} & C^p(X; \hat{H}^{p,p})_0 & \xlongequal{\quad} & Z^p(X) \end{array}$$

where f_* and i_* are the usual proper pushouts on cycles. Recall from [Ros96, (R2d)] that $N_{L/E} \circ \varphi_* = [L : E] \cdot Id$ for the cycle module $\hat{H}^{1,1}$. Thus, $N_{L/E}$ is surjective. As algebraically equivalent cycles are stable by direct images of cycles, we are reduced to the case of the scheme Y' , in codimension 1, already obtained above. \square

Remark 4.13. In the proof, if we can replace the alteration by a (proper birational) resolution of singularities, then the theorem is true with integral coefficients. This is the case in characteristic 0 but also for surfaces and 3-dimensional varieties in characteristic p .

4.4. Mixed Weil cohomologies. Consider a presheaf of differential graded K -algebras \mathbb{E} over the category of smooth schemes. For any closed pair (X, Z) and any integer n , we put :

$$H_Z^n(X, \mathbb{E}) = H^n(\text{Cone}(\mathbb{E}(X) \rightarrow \mathbb{E}(X - Z))).$$

Recall from [CD06] that a mixed Weil cohomology theory over k with coefficients in K is a presheaf \mathbb{E} as above satisfying the followings :

- (1) For $X = \text{Spec}(k), \mathbb{A}_k^1, \mathbb{G}_m$,

$$\dim_K H^i(X) = \begin{cases} 1 & \text{if } i = 0 \text{ or } (X = \mathbb{G}_m, i = 1) \\ 0 & \text{otherwise} \end{cases}$$

- (2) For any excisive morphism $(Y, T) \rightarrow (X, Z)$, the induced morphism $H_Z^*(X, \mathbb{E}) \rightarrow H_T^*(Y, \mathbb{E})$ is an isomorphism.
(3) For any smooth schemes X, Y , the exterior cup-product induces an isomorphism

$$\bigoplus_{p+q=n} H^p(X, \mathbb{E}) \otimes_K H^q(X, \mathbb{E}) \rightarrow H^n(X \times Y, \mathbb{E}).$$

It is proven in [CD06, 2.7.11] that there is a covariant symmetric monoidal triangulated functor

$$R_{\mathbb{E}} : DM_{gm}(k) \rightarrow D^b(K)$$

such that

$$H : DM_{gm}(k)^{op} \rightarrow K - vs, \mathcal{M} \mapsto H^0(R_H(\mathcal{M}^\vee))$$

extends the cohomological functor $H^*(\cdot, \mathbb{E})$.

The twists on this cohomology theory can be described by defining for any K -vector spaces V :

$$V(n) = \begin{cases} V \otimes_K H^1(\mathbb{G}_m, E)^{\otimes -n} & \text{if } n \geq 0, \\ V \otimes_K \text{Hom}_K(H^1(\mathbb{G}_m, E)^{\otimes n}, K) & \text{otherwise.} \end{cases}$$

With these notations, $H(M(X)(-n)[-i]) = H^i(X, \mathbb{E})(n)$. As the functor H is symmetric monoidal, for any smooth projective scheme of dimension n , the morphism $\eta : M(X) \otimes M(X)(-n)[-2n]$ defined in 2.15, induces a perfect pairing, the Poincaré duality pairing,

$$H^i(X, \mathbb{E}) \otimes_K H^{2n-i}(X, \mathbb{E})(n) \rightarrow K, x \otimes y \mapsto p_*(x.y).$$

As in the preceding section, the unit $1 \in H^0(\text{Spec}(k))$ defines a regulator map

$$\sigma^{q,n} : H_{\mathcal{M}}^q(X; \mathbb{Z}(n)) \rightarrow H^q(X, \mathbb{E})(n)$$

compatible with pullbacks, pushouts and products. For any function field L , we deduce a morphism

$$\hat{\sigma}^{q,n} : \bar{H}_{\mathcal{M}}^q(L, \mathbb{Z}(n)) \rightarrow \bar{H}^q(L, \mathbb{E})(n)$$

which is compatible with restriction, norm, residues and products. in other words, we get a canonical morphism of cycle modules $\hat{\sigma}^{q,n} : \hat{H}_{\mathcal{M}}^{q,n} \rightarrow \hat{\mathbb{E}}^{q,n}$. As regulators are “higher cycle classes”, the preceding maps are “higher symbols”. Indeed, we obtain the classical (cohomological) symbol map $K_n^M(L) \rightarrow \bar{H}^n(L, \mathbb{E})(n)$ with $q = n$ above.

Remark 4.14. Given a generator of $H^1(\mathbb{G}_m, \mathbb{E})$, we obtain canonical isomorphisms $H^*(X, \mathbb{E})(n) \simeq H^*(X, \mathbb{E})$ for any integer n . The cycle modules associated to H in the above thus satisfies the following relation : $\hat{H}_*^{q,n} = \hat{H}_{*-q}^{0,n-q} \simeq \hat{H}_{*-q}^{0,0}$.

Corollary 4.15. *Consider a mixed Weil cohomology \mathbb{E} with the notations above. Let $\mathcal{H}^p(\mathbb{E})$ be the Zariski sheaf associated with $H^p(\cdot, \mathbb{E})$.*

Assume that for any function field L/k and any negative integer i , $\bar{H}^i(L, \mathbb{E}) = 0$. Then, the following conditions are equivalent :

- (i) *For any function field L , $\bar{H}^0(L, \mathbb{E}) = K$.*
- (ii) *For any integer $p \in \mathbb{N}$ and any projective smooth scheme X , the regulator map $\sigma^{p,p} : H_{\mathcal{M}}^p(\cdot; \mathbb{Z}(p)) \rightarrow H^p(\cdot, \mathbb{E})(p)$ induces a canonical isomorphism*

$$A^p(X)_K \rightarrow H_{\text{Zar}}^p(X; \mathcal{H}^p(\mathbb{E}))(p).$$

Proof. Remark that the assumption implies that for any smooth scheme X and any $i < 0$, $H^i(X, \mathbb{E}) = 0$ – apply the coniveau spectral sequence for X .

(i) \Rightarrow (ii) : We apply the proposition 4.12 together with remark 4.11. Assumption (Vanishing) and (Rigidity) are among our hypothesis. Remark the (Rigidity) assumption and the Poincaré duality pairing implies that for any smooth projective connected curve $p : C \rightarrow \text{Spec}(k)$, the morphism $p_* : H^2(C, \mathbb{E})(1) \rightarrow H^0(C) = K$ is an isomorphism. Following classical arguments, this together with the multiplicativity of the cycle class map implies that homological equivalence for \mathbb{E} is between rational and numerical equivalence. From Matsusaka's theorem ([Mat57]), these two latter equivalences coincide for divisors. This implies assumption (i) of proposition 4.12.

(ii) \Rightarrow (i) : For a d -dimensional smooth projective connected scheme X , we deduce from the coniveau spectral sequence and Poincaré duality that $E_2^{d,d}(X, d) = H^{2d}(X, \mathbb{E})(d) = H^0(X, \mathbb{E})$. Thus, we obtain $H^0(X, \mathbb{E}) = K$. If L is the function field of X , we deduce that $\bar{H}^0(L, \mathbb{E}) = K$. Considering any function field E , we construct easily an irreducible projective scheme X over k with function field E . Applying De Jong's theorem, we find an alteration $\tilde{X} \rightarrow X$ such that \tilde{X} is projective smooth and the function field L of \tilde{X} is a finite extension of E and the result now follows from the fact $N_{L/E} : \bar{H}^0(L) \rightarrow \bar{H}^0(E)$ is a split epimorphism. \square

Example 4.16. Condition (i) in the previous corollary is only reasonable when the base field k is separably closed⁷.

Assume k is a separably closed field of exponential characteristic p . Condition (i) above is fulfilled by the following mixed Weil cohomology theories : algebraic De Rham cohomology if $p = 0$, rational étale l -adic cohomology if $p \neq l$, rigid cohomology (k is the residue field of a complete valuation ring with field of fraction K). The case of rigid cohomology was in fact our motivation.

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⁷The fact that condition (ii) in the corollary is satisfied for algebraic De Rham cohomology for any field k of characteristic 0 is inadvertently asserted in [BO74, (7.6)]. It is obviously false as we can see in the case of $X = \text{Spec}(E)$ where E/k is a non trivial finite extension.

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